Introduction to Modal Logic

Study Notes

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1 Preliminary Notions

Definition 1.1 (Relational structures)**.** A *relational structure* is a pair (W, R) where W is a nonempty set and $R \subseteq W \times W$ is a binary relation.

Given a binary relation R, we can obtain its *transitive closure*

$$R^+ = \bigcap\{R' \mid R' \subseteq W^2 \text{ is transitive and } R \subseteq R'\}$$

or its *reflexive transitive closure*

$$R^* = \bigcap\{R' \mid R' \subseteq W^2 \text{ is reflexive and transitive and } R \subseteq R'\}$$

Definition 1.2 (Tree)**.** A *tree* is a relational structure (T, S) satisfying:

(i) There is a unique root $r \in T$ such that for all $t \in T$, rS^*t.

(ii) Every element other than the root has a unique predecessor.

(iii) S is acyclic.

Definition 1.3 (Basic modal language)**.** The *basic modal language* is defined by extending the language of propositional logic according to the following grammar, where p ranges over a set $\Phi = \{p, q, r, \dots\}$ of propositional letters:

$$\varphi ::= p \mid \bot \mid \neg\varphi \mid \varphi \vee \varphi \mid \Diamond\varphi$$

We abbreviate $\Box\varphi = \neg\Diamond\neg\varphi$, and the constant $\top$ and the binary connectives $\wedge, \rightarrow, \leftrightarrow$ are also abbreviations, as in usual presentations of propositional logic.

Definition 1.4 (Kripke frames and Kripke models)**.** A *(Kripke) frame* is a relational structure $\mathfrak{F} = (W, R)$. A frame $\mathfrak{F}$ can be combined with a *valuation* function $V : \Phi \rightarrow \mathcal{P}(W)$ to create a *(Kripke) model* $\mathfrak{M} = (\mathfrak{F}, V) = (W, R, V)$.

Definition 1.5 (Satisfaction and validity)**.** Given a model $\mathfrak{M} = (W, R, V)$ and $w \in W$, we say that a formula φ in the basic modal language is *satisfied* in $\mathfrak{M}$ at w, written $\mathfrak{M}, w \Vdash \varphi$, according to the following conditions:

$\mathfrak{M}, w \nVdash \bot$

$\mathfrak{M}, w \Vdash p$ iff $w \in V(p)$

$\mathfrak{M}, w \Vdash \neg\varphi$ iff $\mathfrak{M} \nVdash \varphi$

$\mathfrak{M}, w \Vdash \varphi \vee \psi$ iff $\mathfrak{M}, w \Vdash \varphi$ or $\mathfrak{M}, w \Vdash \psi$

$\mathfrak{M}, w \Vdash \Diamond\varphi$ iff there exists a $w' \in W$ such that wRw' and $\mathfrak{M}, w' \Vdash \varphi$

$\mathfrak{M}, w \Vdash \Box\varphi$ iff for all $w' \in W$, if wRw', then $\mathfrak{M}, w' \Vdash \varphi$

Whenever $\mathfrak{M}, w \Vdash \varphi$ for every $w \in W$ we can simply write $\mathfrak{M} \Vdash \varphi$. If this happens not only for every point but for every valuation V, we say that the frame $\mathfrak{F}$ underlying $\mathfrak{M}$ *validates* φ, written $\mathfrak{F} \Vdash \varphi$.

2 Models

2.1 Invariance results

Definition 2.1 (Modal equivalence)**.** Given Kripke models $\mathfrak{M} = (W, R, V)$ and $\mathfrak{M}' = (W', R', V')$, we say that two points $w \in W$ and $w' \in W'$ are *modally equivalent*, written $w \leftrightsquigarrow w'$, if for all basic modal formula φ, $\mathfrak{M}, w \Vdash \varphi$ if and only if $\mathfrak{M}', w' \Vdash \varphi$.

Definition 2.2 (Disjoint union)**.** Let I be a set of indices, and let $\mathfrak{M}_i = (V_i, R_i, V_i)$ denote a Kripke model for every $i \in I$. We assume that the $\mathfrak{M}_i$ are all disjoint (they do not share any elements between their domains or relations). The *disjoint union* $\biguplus_{i \in I} \mathfrak{M}_i = (W, R, V)$ is defined as $W = \bigcup_{i \in I} W_i$, $R = \bigcup_{i \in I} R_i$ and for every $p \in \Phi$, $V(p) = \bigcup_{i \in I} V_i(p)$.

Proposition 2.1 (Invariance of modal equivalence under disjoint unions)**.** *Let $\{\mathfrak{M}_i \mid i \in I\}$ be a collection of disjoint models. For every basic modal formula φ it holds that for every $i \in I$ and $w \in W_i$,*

$$\mathfrak{M}_i, w \Vdash \varphi \text{ if and only if } \biguplus_{i \in I} \mathfrak{M}_i, w \Vdash \varphi$$

Definition 2.3 (Submodels)**.** Let $\mathfrak{M} = (W, R, V)$ and $\mathfrak{M}' = (W', R', V')$ be Kripke models. We say $\mathfrak{M}'$ is a *submodel* of $\mathfrak{M}$ if $W' \subseteq W$, $R' = R \cap (W' \times W')$ and $V'(p) = V(p) \cap W'$ for every $p \in \Phi$.

If it holds that whenever $w \in W'$ and wRv we also have $v \in W'$ (i.e. W' is closed under reachability for R), then we say that $\mathfrak{M}'$ is a *generated submodel* of $\mathfrak{M}$.

Moreover, given $X \subseteq W$, the *submodel generated by* X is the smallest submodel $\mathfrak{M}'$ of $\mathfrak{M}$ such that $X \subseteq W'$ and $\mathfrak{M}'$ is a generated submodel. A *rooted* model is a model generated by a singleton set.

Proposition 2.2 (Invariance of modal equivalence under generated submodels)**.** *If $\mathfrak{M}' = (W', R', V')$ is a generated submodel of $\mathfrak{M} = (W, R, V)$, then for every basic modal formula φ and every point $w \in W'$,*

$$\mathfrak{M}, w \Vdash \varphi \text{ if and only if } \mathfrak{M}', w \Vdash \varphi$$

Definition 2.4 (Bounded morphisms)**.** Let $\mathfrak{M} = (W, R, V)$ and $\mathfrak{M}' = (W', R', V')$ be Kripke models, and let $f : W \to W'$ be a function. We say that f is a *bounded morphism* if for every $w \in W$, the following holds:

(i) The same propositional letters are satisfied at w and $f(w)$.

(ii) If wRv, then $f(w)R'f(v)$.

(iii) If $f(w)R'v'$, then there exists a $v \in W$ such that $f(v) = v'$ and wRv.

If f is surjective we say that $\mathfrak{M}'$ is the *bounded morphic image* of $\mathfrak{M}$.

Proposition 2.3 (Invariance of modal equivalence under bounded morphic images)**.** *Modal satisfaction is invariant under bounded morphisms, i.e. if $\mathfrak{M} = (W, R, V)$ and $\mathfrak{M}' = (W', R', V')$ are Kripke models and f is a bounded morphism between them, then for every $w \in W$ and every basic modal formula φ,*

$$\mathfrak{M}, w \Vdash \varphi \textit{ if and only if } \mathfrak{M}', f(w) \Vdash \varphi$$

Definition 2.5 (Unravelling)**.** Let $\mathfrak{F} = (W, R)$ be a frame generated by $w \in W$. The *unravelling* of $\mathfrak{F}$ around $w \in W$ is the frame $\vec{\mathfrak{F}} = (\vec{W}, \vec{R})$ where:

(i) The set $\vec{W}$ contains all finite sequences $(w, w_1, w_2, \ldots, w_n) \in W^{n+1}$ for all n such that $wRw_1Rw_2 \ldots w_{n-1}Rw_n$.

(ii) If $\vec{s_1}, \vec{s_2} \in \vec{W}$, then if there is a $v \in W$ such that $\vec{s_1} \cdot v = \vec{s_2}$, then $\vec{s_1}\vec{R}\vec{s_2}$.

Additionally, if $\mathfrak{M} = (\mathfrak{F}, V)$ is a model, then we define, for all $p \in \Phi$, $\vec{V}(p) = \{(w_0, \ldots, w_n) \mid w_n \in V(p)\}$.

Proposition 2.4. *If $\vec{\mathfrak{M}}$ is the unravelling around w of $\mathfrak{M}$, then $\mathfrak{M}$ is a bounded morphic image of $\vec{\mathfrak{M}}$.*

2.2 Bisimulations

Definition 2.6 (Bisimulation)**.** Let $\mathfrak{M} = (W, R, V)$ and $\mathfrak{M}' = (W', R', V')$ be Kripke models. A relation $Z \subseteq W \times W'$ is a *bisimulation* if the following conditions are satisfied:

(i) If wZw', then w and w' satisfy the same propositional letters.

(ii) The *forth condition*: if wZw' and wRv, then there exists a $v' \in W'$ such that $w'R'v'$ and vZv'.

(iii) The *back condition*: if wZw' and $w'R'v'$, then there exists a $v \in W$ such that wRv and vZv'.

If two points $w \in W$ and $w' \in W'$ are linked by a bisimulation Z, we say that they are *bisimilar* and write $\mathfrak{M}, w \leftrightarroweq \mathfrak{M}', w'$.

In relation to the invariance results from the previous subsection, we have the following proposition.

Proposition 2.5.

(i) $\mathfrak{M}_i, w \leftrightarroweq \biguplus_{i \in I} \mathfrak{M}_i, w$

(i) If $\mathfrak{M}'$ *is a generated submodel of* $\mathfrak{M}$*, then* $\mathfrak{M}', w \leftrightarroweq \mathfrak{M}, w$.

(iii) If $\mathfrak{M}'$ *is the bounded morphic image of of* $\mathfrak{M}$*, then* $\mathfrak{M}, w \leftrightarroweq \mathfrak{M}', f(w)$.

Theorem 2.6 (The Bisimulation Theorem)**.** *Let* $\mathfrak{M} = (W, R, V)$ *and* $\mathfrak{M}' = (W', R', V')$ *be Kripke models, and let* Z *be a bisimulation between them. For every* $w \in W$*,* $w' \in W'$*, if* $\mathfrak{M}, w \leftrightarroweq \mathfrak{M}', w'$*, then* $w \leftrightsquigarrow w'$.

The converse of the Bisimulation Theorem (modally equivalent points are bisimilar) is not true in general, but it holds whenever the relations R and R' are *image-finite*: the set $R[w] = \{v \in W \mid wRv\}$ is finite.

Theorem 2.7 (Henessy-Milner)**.** *If* $\mathfrak{M}$ *and* $\mathfrak{M}'$ *are image finite, then* $w \leftrightsquigarrow w'$ *if and only if* $\mathfrak{M}, w \leftrightarroweq \mathfrak{M}', w'$.

2.3 Filtrations and the finite model property

Definition 2.7 (Closure under subformulas)**.** A set of formulas Σ is *closed under subformulas* if whenever $\varphi \in \Sigma$, then the subformulas of φ are also in Σ (i.e. $\mathrm{sf}(\varphi) \subseteq \Sigma$).

If $\mathfrak{M} = (W, R, V)$ and $\mathfrak{M}' = (W', R', V')$ are Kripke models, then we can define a relation $\leftrightsquigarrow_\Sigma \subseteq W \times W'$ as follows: $w \leftrightsquigarrow_\Sigma w'$ if and only if, for every $\varphi \in \Sigma$, $\mathfrak{M}, w \Vdash \varphi$ iff $\mathfrak{M}', w' \Vdash \varphi$.

Definition 2.8 (Filtrations)**.** A *filtration* of $\mathfrak{M} = (W, R, V)$ through a subformula-closed Σ is a new model $\mathfrak{M}^f = (W^f, R^f, V^f)$ such that

(i) $W^f = W/_{\leftrightsquigarrow_\Sigma}$

(ii) If wRv, then $[w]R^f[v]$.

(iii) If $[w]R^f[v]$, then for all $\Diamond\varphi \in \Sigma$, if $\mathfrak{M}, v \Vdash \varphi$, then $\mathfrak{M}, w \Vdash \Diamond\varphi$.

(iv) For every $p \in \Phi$, $V^f(p) = \{[w] \mid w \in V(p)\}$.

Theorem 2.8 (The Filtration Theorem)**.** *If $\mathfrak{M}^f = (W^f, R^f, V^f)$ is the filtration of a model $\mathfrak{M} = (W, R, V)$ through some subformula-closed set Σ, then for every $w \in W$, $w \leftrightsquigarrow [w]$.*

Lemma 2.9 (Existence and properties of filtrations)**.** *Relations satisfying the conditions on Definition 2.8 exist. The following relations R^s and R^l are both filtrations:*

$$[w]R^s[v] \text{ iff } \exists w' \in [w], v' \in [v] : w'Rv'$$
$$[w]R^l[v] \text{ iff } \forall \Diamond\varphi \in \Sigma : \mathfrak{M}, v \Vdash \varphi \Rightarrow \mathfrak{M}, w \Vdash \Diamond\varphi$$

Moreover, R^s and R^l are, respectively, the smallest and largest filtrations: for every filtration R^f, $R^s \subseteq R^f \subseteq R^l$.

Besides, every filtration preserves reflexivity. The smallest filtration R^f preserves symmetry and the relation R^t preserves transitivity, where R^t is defined as follows:

$$[w]R^t[v] \text{ iff } \forall \Diamond\varphi \in \Sigma : \mathfrak{M}, v \Vdash \varphi \vee \Diamond\varphi \Rightarrow \mathfrak{M}, w \Vdash \Diamond\varphi$$

Note that if Σ is finite (say $|\Sigma| = n \in \mathbb{N}$), then the set W_Σ is finite, even when W is infinite! Moreover, $|W_\Sigma| \leq 2^n$. By taking $\Sigma = \text{sf}(\varphi)$, we can make φ satisfiable in a finite model.

Theorem 2.10 (Finite Model Property)**.** *Let φ be a basic modal formula. If φ is satisfiable, then it is also satisfiable in a finite model. Moreover, it is satisfiable in a model with at most 2^n points, where n is the number of subformulas of φ.*

2.4 The Standard Translation and modal correspondence

Definition 2.9 (Standard Translation)**.** Let $\mathcal{L}^1$ be the first-order language that has a binary predicate R and, for every $p \in \Phi$, a unary predicate P. Then, given a first-order variable x and a basic modal formula φ, we define the *standard translation* of φ into $\mathcal{L}^1$, $\mathsf{ST}_x(\varphi)$, as follows:

$$\begin{aligned}
\mathsf{ST}_x(\bot) &= x \neq x \\
\mathsf{ST}_x(p) &= Px \\
\mathsf{ST}_x(\neg\varphi) &= \neg\,\mathsf{ST}_x(\varphi) \\
\mathsf{ST}_x(\varphi \vee \psi) &= \mathsf{ST}_x(\varphi) \vee \mathsf{ST}_x(\psi) \\
\mathsf{ST}_x(\Diamond\varphi) &= \exists y(Rxy \wedge \mathsf{ST}_y(\varphi)) \\
\mathsf{ST}_x(\Box\varphi) &= \forall y(Rxy \to \mathsf{ST}_y(\varphi))
\end{aligned}$$

Note that a Kripke model $\mathfrak{M}$ can be easily seen as a first-order model in $\mathcal{L}^1$: W is the domain, R determines how to interpret the binary relation symbol, and V determines how to interpret the unary predicates. Hence, allowing some abuse of notation, we have the following proposition.

Proposition 2.11. *Let $\mathfrak{M} = (W, R, V)$ be a Kripke model, let $w \in W$ and let φ be a basic modal formula. The following holds:*

(i) $\mathfrak{M}, w \Vdash \varphi$ *if and only if* $\mathfrak{M} \models \mathsf{ST}_x(\varphi)[w]$

(ii) $\mathfrak{M} \Vdash \varphi$ *if and only if* $\mathfrak{M} \models \forall x\,\mathsf{ST}_x(\varphi)$

Definition 2.10. A well-formed formula $\alpha(x)$ in the language $\mathcal{L}^1$ is said to be *invariant under bisimulations* if for any bisimulation Z between $\mathfrak{M} = (W, R, V)$ and $\mathfrak{M}' = (W', R', V')$ and every pair of points $w \in W$, $w' \in W'$ such that $\mathfrak{M}, w \leftrightarroweq \mathfrak{M}', w'$, it holds that $\mathfrak{M} \models \alpha(x)[w]$ if and only if $\mathfrak{M}' \models \alpha(x)[w']$.

Theorem 2.12 (Van Benthem's Characterization Theorem)**.** *A first-order formula $\alpha(x)$ in $\mathcal{L}^1$ is invariant under bisimulation if and only if $\alpha(x)$ is equivalent to $\mathsf{ST}_x(\varphi)$ for some basic modal formula φ.*

Corollary 2.13. *The basic modal language is a bisimulation-invariant fragment of first-order logic.*

3 Frames

3.1 Invariance results

Proposition 3.1 (Invariance results for frames)**.** *Disjoint unions, generated subframes, and bounded morphisms on frames preserve modal validity:*

(i) *If* $\mathfrak{F}_i \Vdash \varphi$ *for all* $i \in I$*, then* $\biguplus_{i\in U} \mathfrak{F}_i \Vdash \varphi$.

(ii) *If* $\mathfrak{F}'$ *is a generated subframe of* $\mathfrak{F}$*, then* $\mathfrak{F} \Vdash \varphi$ *implies* $\mathfrak{F}' \Vdash \varphi$.

(iii) *If* $\mathfrak{F}'$ *is the bounbed morphic image of* $\mathfrak{F}$*, then* $\mathfrak{F} \Vdash \varphi$ *implies* $\mathfrak{F}' \Vdash \varphi$.

Proposition 3.2. *Consider a Kripke frame* $\mathfrak{F}$*, a basic modal formula* φ *on* n *propositional letters* $p_1, \ldots, p_n$ *and a second-order language* $\mathcal{L}^2$ *with one binary predicate* R*. Then,*

(i) $\mathfrak{F}, w \Vdash \varphi$ *if and only if* $\mathfrak{F} \models \forall P_1 \ldots \forall P_n \, \mathsf{ST}_x(\varphi)[w]$

(ii) $\mathfrak{F} \Vdash \varphi$ *if and only if* $\mathfrak{F} \models \forall x \forall P_1 \ldots \forall P_n \, \mathsf{ST}_x(\varphi)$

3.2 Definability and frame correspondents

Definition 3.1 (Frame definability and frame correspondents)**.** A modal formula φ *defines* a class of frames $\mathcal{C}$ if for every frame $\mathfrak{F}$, $\mathfrak{F} \in \mathcal{C}$ if and only if $\mathfrak{F} \Vdash \varphi$.

Sometimes, the property defining the class $\mathcal{C}$ can be expressed in a first-order formula. We say that a first-order formula α defines a class of frames $\mathcal{C}$ if for every frame $\mathfrak{F}$, we have $\mathfrak{F} \in \mathcal{C}$ if and only if $\mathfrak{F} \models \alpha$.

If a modal formula φ and a first-order formula α define the same class, we call them *frame correspondents*.

Proposition 3.3 (Modal formulas that define classes)**.**

(i) *The formula* $\Box p \to p$ *defines the class of reflexive frames.*

(ii) *The formula* $\Diamond\Box p \to p$ *defines the class of symmetric frames.*

(iii) *The formula* $\Diamond\Diamond p \to \Diamond p$ *defines the class of transitive frames.*

(iv) *Löb's formula* $\Box(\Box p \to p) \to \Box p$ *defines the class of trasnitive frames without infinite sequences.*

(v) *Grzegorczyk's formula* $\Box(\Box(p \to \Box p) \to p) \to p$ *defines the class of reflexive, transitive frames without nontrivial infinite sequences.*

Definition 3.2 (Sahlqvist formulas)**.** The formula $\Diamond^n p$ for $n \in \mathbb{N}$ and $p \in \Phi$ is called a *diamonded* atom. Analogously, the formula $\Box^n p$ for $n \in \mathbb{N}$ and $p \in \Phi$ is called a *boxed* atom.

A formula is called *positive* if all occurrences of $p \in \Phi$ are in the scope of an even number of negations.

A *Sahlqvist ancecedent* is built from $\bot, \top$, boxed atoms and negative formulas using $\wedge$ and$\Diamond$.

A formula is called *negative* if all occurrences of $p \in \Phi$ are in the scope of an odd number of negations.

A *Sahlqvist implication* is a formula $\varphi \to \psi$ where φ is a Sahlqvist antecedent and ψ is positive.

A *Sahlqvist formula* is built built from Sahlqvist implications using $\wedge$ and $\Box$.

Theorem 3.4 (Sahlqvist's Correspondence Theorem)**.** *For all Sahlqvist formula, there exists a first-order frame correspondent.*

Algorithm 3.1 (Sahlvist-Van Benthem)**.** A *simple Sahlqvist formula* uses only $\bot, \top$ and boxed atoms in its antecedents. For a simple Sahlqvist formula φ we can use the Sahlvist-Van Benthem algorithm to find a first-order correspondent:

1. Identify boxed atoms in the antecedent.
2. Draw the picture that discusses the minimal valuation that makes the antecedent true. Name the worlds involved by $t_0, \ldots, t_n$.
3. Work out the minimal valuation: get a first-order expression for it in terms of the named worlds.
4. Work out the standard translation of φ. Use the names you fixed for the variables that correspond to $\Diamond$ in the antecedent.
5. Pull out the quantifiers that bind the t_i variables in the antecedent to the front. For this use the equivalences
$$\exists x \alpha(x) \wedge \beta \leftrightarrow \exists x(\alpha(x) \wedge \beta)$$
$$\exists x \alpha(x) \to \beta \leftrightarrow \forall x(\alpha(x) \to \beta)$$
6. Replace all the predicates $P(x), Q(x), \ldots$ with the first-order expression corresponding to the minimal valuation.
7. Simplify, if possible.
8. Add $\forall x$ (binding the free variable of the standard translation) to the resulting first-order formula to obtain the global first-order correspondent.

4 Normal Modal Logics

Let us fix a class $\mathcal{C}$ of frames. We need to have, whenever possible, an effective criterion (algorithm) deducing whether a formula φ is valid in $\mathcal{C}$.

If $\mathcal{C}$ is infinite then going through all the frames might take infinite time. Even if $\mathcal{C}$ is finite, but contains an infinite frame, the procedure might still take infinite time.

In order to overcome this difficulty we will develop a syntactic (axiomatic) approach to modal logic. The idea of this approach is to find a small (possibly finite) number of formulas (axioms of our logic) and set some rules of inference which enable us to derive other formulas (theorems of our logic).

We start by defining the class of formulas valid in a class of frames: its logic.

Definition 4.1 (Logic of a class of frames)**.** Let $\mathcal{C}$ be a class of frames. We define the *logic of* $\mathcal{C}$ as the set of formulas that are valid in all frames of $\mathcal{C}$:

$$\mathrm{Log}(\mathcal{C}) = \{\varphi \mid \mathfrak{F} \Vdash \varphi \text{ for every } \mathfrak{F} \in \mathcal{C}\}$$

Whenever $\mathcal{C} = \{\mathfrak{F}\}$, a set with a single frame, we can simply write $\mathrm{Log}(\mathfrak{F})$ instead of $\mathrm{Log}(\{\mathfrak{F}\})$. We denote by $\mathrm{Log}(\emptyset)$ the set of all formulas: the inconsistent logic.

Proposition 4.1. *Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be classes of frames. If $\mathcal{C}_1 \subseteq \mathcal{C}_2$, then $\mathrm{Log}(\mathcal{C}_2) \subseteq \mathrm{Log}(\mathcal{C}_1)$.*

Theorem 4.2 (Makinson's theorem of 'maximal' logics)**.** *Let $\mathcal{C}$ be a non-empty class of frames. Then $\mathrm{Log}(\mathcal{C})$ is contained in the logic of a single reflexive point or $\mathrm{Log}(\mathcal{C})$ is contained in the logic of a single irreflexive point.*

Definition 4.2 (Normal modal logic)**.** A *normal modal logic* L is a set of formulas that contains all propositional tautologies, the axioms

$$\Box(p \to q) \to (\Box p \to \Box q) \tag{K}$$

$$\Diamond p \leftrightarrow \neg\Box\neg p \tag{Dual}$$

and is closed under the rules of *modus ponens* (MP), Necessitation (Nec) and Uniform Substitution (US):

$$\frac{\varphi \qquad \varphi \to \psi}{\psi}\,(\mathrm{MP}) \qquad \frac{\varphi}{\Box\varphi}\,(\mathrm{Nec}) \qquad \frac{\varphi(p_1, \ldots, p_n)}{\varphi(\psi_1, \ldots, \psi_n)}\,(\mathrm{US})$$

Proposition 4.3 (Validity of normal axioms and rules). *The axioms K and Dual are valid in all frames. Similarly, the rules (MP), (Nec) and (US) preserve validity on all frames.*

Proposition 4.4. *For every class of frames $\mathcal{C}$,* $\mathrm{Log}(\mathcal{C})$ *is a normal modal logic.*

Definition 4.3 (**K**). We denote by **K** the smallest normal modal logic i.e. the smallest set of formulas containing all propositional tautologies, containing axioms K and Dual and closed under the rules of (MP), (Nec) and (US).

Notation 4.4. Let L be a modal logic. Instead of $\varphi \in L$, we often write $\vdash_L \varphi$, read as "φ is a theorem of L".

4.1 Soundness and completeness

Definition 4.5 (Soundness and completeness). Let L be a logic and let $\mathcal{C}$ be a class of frames. If $L \subseteq \mathrm{Log}(\mathcal{C})$, we say that L *is sound with respect to* $\mathcal{C}$. That is, if $\vdash_L \varphi$, then $\mathcal{C} \Vdash \varphi$.

On the other hand, if $\mathrm{Log}(\mathcal{C}) \subseteq L$, we say that L *is complete with respect to* $\mathcal{C}$. That is, if $\mathcal{C} \Vdash \varphi$, then $\vdash_L \varphi$.

Together, whenever a logic L is *sound and complete* with respecto to $\mathcal{C}$, we have that for every formula φ,

$$\vdash_L \varphi \text{ if and only if } \mathcal{C} \Vdash \varphi$$

In what follows we outline the main results leading to the soundness and completeness of **K** with respect to the class $\mathcal{A}$ of all frames.

Soundness follows immediately from Proposition 4.3: all the axioms in **K** are valid on all frames, and all rules preserve validity, so any formula in **K** must be valid on all frames.

Corollary 4.5 (Soundness for **K**). *Every theorem of* **K** *is valid in every Kripke frame:* $\mathbf{K} \subseteq \mathrm{Log}(\mathcal{A})$.

Completeness is more difficult. We use the *canonical model construction*, which will allow us to show that every consistent set of formulas is satisfiable. We first define these notions and make them more rigorous.

Definition 4.6 (L-consistency). Let L be a normal modal logic and let Γ be a set of formulas. We say that Γ is *L-consistent* if $\Gamma \nvdash_L \bot$, and *L-inconsistent*, otherwise. Here, $\Gamma \vdash_L \varphi$ means that there are formulas $\gamma_1, \ldots, \gamma_n \in \Gamma$ such that $\vdash_L (\gamma_1 \wedge \cdots \wedge \gamma_n) \to \varphi$. In what follows, *consistent* means **K**-consistent.

Definition 4.7 (Satisfiability)**.** A set Γ of formulas is said to be satisfiable if there exists a Kripke model $\mathfrak{M}$ and point w in this model such that $\mathfrak{M}, w \Vdash \varphi$ for every $\varphi \in \Gamma$.

Proposition 4.6. *If a set of formulas is satisfiable, then it is consistent.*

The previous proposition is an alternative road to soundness, and its converse (every consistent set is satisfiable) will be our road to completeness. But that is more difficult: we need to provide a model! This will be the *canonical model.* In this model, the points will be consistent sets of sentences. In particular, *maximally consistent* sets of sentences.

Definition 4.8 (Maximal consistency)**.** Let L be a logic and Γ be a set of sentences. We say that Γ is *maximally L-consistent* (or simply *maximally consistent*) is Γ is L-consistent and any proper superset of Γ is L-inconsistent.

Lemma 4.7 (Lindenbaum's Lemma)**.** *Every consistent set of sentences Γ can be extended into a maximally consistent set $\hat{\Gamma}$.*

Proposition 4.8 (Properties of maximally consistent sets)**.** *Let Γ be maximally L-consistent. Then, the following hold:*

(i) *At least one of $\Gamma \cup \{\varphi\}$ or $\Gamma \cup \{\neg\varphi\}$ is consistent (this holds too even if Γ is just consistent but not maximally so).*

(ii) *For every φ, either $\varphi \in \Gamma$ or $\neg\varphi \in \Gamma$.*

(iii) *For pair of formulas φ and ψ, $\varphi \vee \psi \in \Gamma$ if and only if $\varphi \in \Gamma$ or $\psi \in \Gamma$.*

(iv) *For pair of formulas φ and ψ, $\varphi \wedge \psi \in \Gamma$ if and only if $\varphi \in \Gamma$ and $\psi \in \Gamma$.*

(v) *For every φ, $\varphi \in \Gamma$ if and only if $\Gamma \vdash \varphi$.*

(vi) $L \subseteq \Gamma$

(vii) Γ *is closed under (MP).*

(viii) Γ *is **not** necessarily closed under (Nec) or (US).*

We are now ready to define the *canonical model.*

Definition 4.9 (Canonical model for **K**)**.** The *canonical model* $\mathfrak{M}^c = (W^c, R^c, V^c)$ is the Kripke model defined as follows:

(i) $W^c = \{\Gamma \mid \Gamma \text{ is a maximally consistent set}\}$

(ii) The relation R^c is such that $\Gamma R^c \Delta$ if and only if for all $\Box\varphi \in \Gamma$, we have $\varphi \in \Delta$ (or, equivalently, for all $\varphi \in \Delta$, we have $\Diamond\varphi \in \Gamma$).

(iii) For every $p \in \Phi$, $V^c(p) = \{\Gamma \in W^c \mid p \in \Gamma\}$.

We denote by $\mathfrak{F}^c = (W^c, R^c)$ the *canonical frame*. In general, for a normal modal logic L, we denote by $\mathfrak{M}^c_L = (W^c_L, R^c_L, V^c_L)$ the canonical model for L.

With the canonical model in hand, we can prove the main result towards completeness: the Truth Lemma.

Lemma 4.9 (Truth Lemma)**.** *For every formula φ and every maximally consistent set Γ,*

$$\mathfrak{M}^c, \Gamma \Vdash \varphi \textit{ if and only if } \varphi \in \Gamma$$

The Truth Lemma lets us prove the Satisfiability theorem (the converse of Proposition 4.6): if a set Γ is consistent, then we extend into a maximally consistent set $\hat{\Gamma} \supseteq \Gamma$ by Lindenbaum's Lemma, and by the Truth Lemma, $\mathfrak{M}^c, \hat{\Gamma} \Vdash \varphi$ for every $\varphi \in \Gamma$.

Proposition 4.10. *Every consistent set is satisfiable.*

We are now ready to state (and prove) completeness: suppose $\mathcal{A} \Vdash \varphi$. Then $\{\neg\varphi\}$ is not satisfiable in $\mathcal{A}$. By the contrapositive of the previous proposition we have that $\{\neg\varphi\}$ is inconsistent: $\{\neg\varphi\} \vdash \bot$; that is, $\vdash \neg\varphi \to \bot$. By propositional tautologies, $\vdash \neg\neg\varphi$, and by propositional tautologies again, $\vdash \varphi$. This was to show.

Theorem 4.11 (Completeness for **K**)**.** *The logic* **K** *is complete with respect to the class $\mathcal{A}$ of all frames:* $\text{Log}(\mathcal{A}) \subseteq \mathbf{K}$.

Remark 4.1. The previous method can be generalized to other logics. To every logic? Well, no. The canonical model technique lets us prove Kripke-completeness (completeness with regards to a class of Kripke frames), but there exist consistent logics that are not Kripke-complete!

We say that a logic is *canonical* when the canonical frame $\mathfrak{F}^c_L$ validates every formula in the logic.

For canonical logics, we can prove Kripke-completeness in a very straightforward way. To show completeness with respect to a class $\mathcal{C}$, simply show that the relation R^c_L has the properties needed for a frame to be in $\mathcal{C}$.

On the other hand, there exist logics that are complete but not canonical, as well as logics that are not complete. For these, the method will not be very useful.

Remark 4.2 (Other normal modal logics). If we take **K** and add extra axioms (formulas defining certain properties), we will get logics that characterize the logic of a certain class of frames. Consider, for example, the following axioms:

$$\Diamond\Diamond p \to p \text{ (equivalent to } \Box p \to \Box\Box p) \tag{4}$$

$$p \to \Diamond p \text{ (equivalent to } \Box p \to p) \tag{T}$$

$$p \to \Box\Diamond p \tag{D}$$

$$\Diamond\Box p \to \Box\Diamond p \tag{.2}$$

$$\Box(\Box p \to q) \vee \Box(\Box q \to p) \tag{.3}$$

$$\Box(\Box p \to p) \to \Box p \tag{L}$$

The following are some some logics using the previous axioms, followed by the properties they define.

Logic	*Class of frames*
K	the class of all frames
K4	the class of transitive frames
T	the class of reflexive frames
B	the class of symmetric frames
KD	the class of right-unbounded frames
S4	the class of reflexive, transitive frames
S5	the class of frames whose relation is an equivalence relation
K4.3	the class of transitive frames with no branching to the right
S4.2	the class of reflexive, transitive and directed frames
S4.3	the class of reflexive, transitive frames with no branching to the right
KL	the class of finite transitive trees (weak completeness only)

4.2 Decidability and the finite model property

Suppose we proved that a normal modal logic L is complete with respect to a class $\mathcal{C}$ of Kripke frames; that is $L = \text{Log}(C)$. The class $\mathcal{C}$ might be huge. It might consist of infinitely many infinite frames. So this still does not give us a criterion for deciding whether a given formula φ belongs to L. However, if we manage to show a stronger version of completeness — a completeness with respect to a class of finite frames— then we are one step closer to having such a criterion.

Definition 4.10 (Finite model property)**.** We say that a normal modal logic L has the *finite model property* if there exists a (not necessarily finite) class $\mathcal{C}$ of finite frames such that $L = \text{Log}(\mathcal{C})$.

If we already know that a logic L is Kripke-complete with respect to a class $\mathcal{C}$, we can try an prove completeness with respecto the class $\mathcal{C}_{\text{fin}} \subseteq \mathcal{C}$ of finite frames in $\mathcal{C}$. To do so, we can take the filtration of the canonical model in such a way that the properties of the frames in $\mathcal{C}$ (say, transitivity) are preserved.

If a logic L has the finite model property, then we can easily devise a procedure that can decide, for every formula φ, whether $\mathcal{C} \Vdash \varphi$. On the one hand, list all theorems by means of mechanically deriving all proofs in the logic. If φ is valid, then it will eventually show up in that list. Simulatenously, we can list all non-theorems by evaluating the formula at every finite frame of its class. If φ is not valid, then eventually there will be a finite model where it will be refuted.

This argument is captured in the following theorem, which can be invoked to justify the finite model property for usual logics.

Theorem 4.12 (Harrop's Theorem)**.** *If a logic L has the finite model property and it is finitely axiomatizable, then L is decidable.*

4.3 Incompleteness results

Though cononicity proofs are powerful, there are continuum many Kripke-incomplete logics! As a taste of this, we will show that the basic temporal logic is incomplete. We first need the notion of general frames.

4.3.1 General frames

Definition 4.11 (General frame)**.** A triple $\mathfrak{G} = (W, R, A)$ is a *general frame* if (W, R) is a Kripke frame and $A \subseteq \mathcal{P}(W)$ verifies

(i) $W \in A$ and $\emptyset \in A$.

(ii) If $U, V \in A$, then $U \cap V \in A$.

(iii) If $U \in A$, then $W \setminus U \in A$.

(iv) If $U \in A$, then $\Diamond_R U \in A$, where $\Diamond_R U = \{x \in W \mid \exists y \in U : xRy\}$.

The set A is called the set of *admissible valuations*, as whenever we work on a general frame, valuations are functions of the form $V : \Phi \to A \subseteq \mathcal{P}(W)$.

Finally, for a normal modal logic L, a general frame is called a L-frame if L is valid on the frame.

Proposition 4.13. *Let $\mathfrak{F} = (W, R)$ be a Kripke frame and let $\mathfrak{G} = (\mathfrak{F}, A)$ be a general frame. If $\mathfrak{F} \Vdash \varphi$, then $\mathfrak{G} \Vdash \varphi$, but the converse is not necessarily true.*

Proposition 4.14. *Let L be a normal modal logic. Then L is sound and complete with respect to the class of general L-frames.*

Proposition 4.15. *Let $\mathfrak{G}$ be a general frame, and let $\mathrm{Log}(\mathfrak{G}) = \{\varphi \mid \mathfrak{G} \Vdash \varphi\}$. The set $\mathrm{Log}(\mathfrak{G})$ is a consistent modal logic.*

4.3.2 Incompleteness of $\mathbf{K}_t\,\mathbf{ThoM}$

Using general frames we can show that consistent but incomplete normal modal logics exist.

We now demonstrate the existence of incomplete logics in the basic temporal language. The demonstration has three main steps. First, we introduce a tense logic called $\mathbf{K}_t\,\mathbf{Tho}$ and show that it is consistent. Second, we show that no frame for $\mathbf{K}_t\,\mathbf{Tho}$ can validate the McKinsey axiom (which in tense logical notation is $GF\varphi \to FG\varphi$). It is tempting to conclude that $\mathbf{K}_t\,\mathbf{Tho}$, the smallest tense logic containing both $\mathbf{K}_t\,\mathbf{Tho}$ and the McKinsey axiom, is the inconsistent logic. Surprisingly, this is not the case. $\mathbf{K}_t\,\mathbf{Tho}$ is consistent — and hence is not the tense logic of any class of frames at all. We prove this in the third step with the help of general frames.

$\mathbf{K}_t\,\mathbf{Tho}$ is the tense logic generated by the following axioms:

$$Fp \wedge Fq \to F(p \wedge Fq) \vee F(p \wedge q) \vee F(Fp \wedge q) \tag{$.3_r$}$$

$$Gp \to Fp \tag{D_r}$$

$$H(Hp \to p) \to Hp \tag{L_l}$$

The first two axioms are canonical for simple first-order conditions (no branching to the right, and right-unboundedness, respectively). The third axiom is simply the Löb axiom written in terms of the backwards looking operator H; it is valid on precisely those frames that are transitive and contain no infinite descending paths. (Note that such frames cannot contain reflexive points.)

Let $\mathbf{K}_t\,\mathbf{Tho}$ be the tense logic generated by these three axioms. As all three axioms are valid on the natural numbers, $\mathbf{K}_t\,\mathbf{Tho}$ is consistent. If (T, R) is a frame for $\mathbf{K}_t\,\mathbf{Tho}$ and $t \in T$, then $\{u \in T \mid tRu\}$ is a right-unbounded strict total order.

Now for the second step. Let $\mathbf{K}_t\,\mathbf{ThoM}$ be the smallest tense logic containing $\mathbf{K}_t\,\mathbf{ThoM}$ and the McKinsey axiom (M), $GFp \to FGp$. What are the frames for this enriched logic? The answer is: none at all, or, to put it another way, $\mathbf{K}_t\,\mathbf{ThoM}$ defines the empty class of frames.

Proposition 4.16. *Let* $\mathfrak{T}$ *be any frame for* $\mathbf{K}_t\,\mathbf{Tho}$. *Then* $\mathfrak{T} \not\Vdash \mathrm{M}$.

We are ready for the final step. As $\mathbf{K}_t\,\mathbf{ThoM}$ defines the empty class of frames, it is tempting to conclude that it is also complete with respect to this class; that is, that $\mathbf{K}_t\,\mathbf{ThoM}$ is the inconsistent logic. However, this is not the case, as there exists a general frame that validates $\mathbf{K}_t\,\mathbf{ThoM}$ (the general frame $(\mathbb{N}, <, A)$, where A is the set of finite and co-finite subsets of $\mathbb{N}$).

Theorem 4.17. $\mathbf{K}_t\,\mathbf{ThoM}$ *is consistent and incomplete.*

4.3.3 Incompleteness of KvB

Let $\mathbf{KvB} = \mathbf{K} + \mathrm{vB}$, where (vB) is the Van Benthem axiom:

$$\Box\Diamond\top \to \Box(\Box(\Box p \to p) \to p) \tag{vB}$$

We can show that **KvB** is a Kripke-incomplete consistent normal modal logic. First we show that there exists a general frame $\mathfrak{G}$, such that $\mathfrak{G} \Vdash \mathrm{vB}$ but $\mathfrak{G} \not\Vdash \Box\Diamond\top \to \Box\bot$. However, it turns out that any Kripke frame that validates (vB) also validates $\Box\Diamond\top \to \Box\bot$.

We can then make the following argument: firstly, $\mathrm{Log}(\mathfrak{G})$ is consistent (it is always the case that the logic of a frame is consistent). Besides, $\mathfrak{G}$ validates everything in $\mathbf{K}$ and, $\mathfrak{G} \Vdash \mathrm{vB}$, so $\mathbf{KvB} \subseteq \mathrm{Log}(\mathfrak{G}) \subsetneq \mathrm{Log}(\emptyset)$: **KvB** is consistent.

Now, assume for a contradiction that the logic **KvB** is Kripke-complete: there exists a class $\mathcal{C}$ of Kripke frames such that $\mathbf{KvB} = \mathrm{Log}(\mathcal{C})$. As $\mathbf{KvB} \subseteq \mathrm{Log}(\mathfrak{G})$, we have $\mathbf{KvB} = \mathrm{Log}(\mathcal{C}) \subseteq \mathrm{Log}(\mathfrak{G})$.

The class $\mathcal{C}$ is nonempty, as otherwise **KvB** would be inconsistent. Hence, $\mathcal{C}$ contains at least one frame. Take $\mathfrak{F} \in \mathcal{C}$. Since $\mathbf{KvB} = \mathrm{Log}(\mathcal{C})$, we have $\mathfrak{F} \Vdash \mathrm{vB}$, and hence $\mathfrak{F} \Vdash \Box\Diamond\top \to \Box\bot$, so $\Box\Diamond\top \to \Box\bot \in \mathrm{Log}(\mathcal{C}) \subseteq \mathrm{Log}(\mathfrak{G})$, so also $\Box\Diamond\top \to \Box\bot \in \mathrm{Log}(\mathfrak{G})$, but this implies $\mathfrak{G} \Vdash \Box\Diamond\top \to \Box\bot$. Contradiction.

We can conclude that there is no class $\mathcal{C}$ of Kripke frames such that $\mathbf{KvB} = \mathrm{Log}(\mathcal{C})$. The logic **KvB** is consistent but Kripke-incomplete.

Theorem 4.18. **KvB** *is a consistent but Kripke-incomplete normal modal logic.*

5 Propositional Dynamic Logic

We now start working on a more powerful modal language: the language of propositional dynamic logic (PDL), where we have possibly infinitely many modalities, each corresponding to a *program*.

Definition 5.1 (The language of PDL)**.** Let Π be a set of programs, and $A \subseteq \Pi$ a set of *atomic* programs. Concretely, Π is the set of programs obtained starting from the set A of atomic programs and operating as follows:

$$\pi ::= \pi \cup \pi \mid \pi ; \pi \mid \pi^*$$

For every $\pi \in \Pi$, we consider a diamond modality $\langle \pi \rangle$. The *language of PDL* is defined by the following grammar, where $\pi \in \Pi$ and $p \in \Phi$:

$$\varphi ::= p \mid \neg\varphi \mid \varphi \vee \varphi \mid \langle \pi \rangle \varphi$$

Definition 5.2 (Regular frames)**.** A frame $\mathfrak{F} = (W, \{R_\pi\}_{\pi \in \Pi})$ for PDL is called *regular* if it holds, for every $\pi, \pi_1, \pi_2 \in \Pi$, that

$$\begin{aligned} R_{\pi_1 \cup \pi_2} &= R_{\pi_1} \cup R_{\pi_2} \\ R_{\pi_1 ; \pi_2} &= R_{\pi_1} \circ R_{\pi_2} \\ R_{\pi^*} &= (R_\pi)^* \end{aligned}$$

We denote by Reg the class of regular frames. A model based on a regular frames is called a *regular model*.

Definition 5.3 (Normal propositional dynamic logics)**.** A normal modal logic L in the language of propositional logic (i.e. a *normal propositional dynamic logic*) is a set of formulas such that it contains all propostional tautologies, axioms

(i) $[\pi](p \to q) \to ([\pi]p \to [\pi]q)$

(ii) $[\pi] \leftrightarrow \neg\langle \pi \rangle \neg p$

(iii) $\langle \pi_1 \cup \pi_2 \rangle p \leftrightarrow \langle \pi_1 \rangle p \vee \langle \pi_2 \rangle p$

(iv) $\langle \pi_1 ; \pi_2 \rangle p \leftrightarrow \langle \pi_1 \rangle \langle \pi_2 \rangle p$

(v) $\langle \pi^* \rangle p \leftrightarrow (p \vee \langle \pi \rangle \langle \pi^* \rangle p)$

(vi) $[\pi^*](p \to [\pi]p) \to (p \to [\pi^*]p)$

and is closed under the usual rules (MP), (Nec) and (US).

We denote by **PDL** the smallest normal propositional dynamic logic.

5.1 Soundness and completeness for PDL

Our goal is now to show that **PDL** is sound and complete with respect to the class Reg of regular frames: **PDL** $= \mathrm{Log}(\mathsf{Reg})$. Soundness follows immediately.

Theorem 5.1 (Soundness of **PDL**)**.** **PDL** *is sound with respect to the class* Reg *of regular frames:* **PDL** $\subseteq \mathrm{Log}(\mathsf{Reg})$*.*

In fact, we could say that **PDL** *defines* the class of regular frames: $\mathfrak{F} \Vdash$ **PDL** if and only if $\mathfrak{F} \in \mathsf{Reg}$.

For completeness, canonicity does not work. For one, **PDL** is not canonical: if it were, then every consistent set would be satisfiable, but this is not the case (the last axioms breaks compactness).

Proposition 5.2. **PDL** *is not canonical.*

Yet, **PDL** is Kripke-complete with respect to Reg, and it even has the finite model property. We show that by creating a finite canonical model for **PDL**. We start by defining a finitary analog to maximally consistent sets.

Definition 5.4 (Fischer-Ladner closure)**.** Let Σ be a set of formulas. Then, Σ is *Fischer-Ladner*-closed (or *FL*-closed, for short) if it is closed under subformulas and satisfies:

(i) If $\langle \pi_1 ; \pi_2 \rangle \varphi \in \Sigma$, then $\langle \pi_1 \rangle \langle \pi_2 \rangle \varphi \in \Sigma$.

(ii) If $\langle \pi_1 \cup \pi_2 \rangle \varphi \in \Sigma$, then $\langle \pi_1 \rangle \varphi \vee \langle \pi_2 \rangle \varphi \in \Sigma$.

(iii) If $\langle \pi^* \rangle \varphi \in \Sigma$, then $\langle \pi \rangle \langle \pi^* \rangle \varphi \in \Sigma$.

We denote by $\mathrm{FL}(\Sigma)$ the smallest set containing Σ that is Fischer-Ladner closed.

Definition 5.5 (Single negation)**.** Given a formula φ, we define the *single negation* of φ, written $\sim \varphi$, as

$$\sim \varphi = \begin{cases} \psi & \text{if } \varphi = \neg \psi \\ \neg \varphi & \text{otherwise} \end{cases}$$

A set Σ is *closed under single negations* if whenever $\varphi \in \Sigma$, we also have $\sim \varphi \in \Sigma$. We denote by $\neg\mathrm{FL}(\Sigma)$ the the smallest set that contains Σ, is Fischer-Ladner-closed and is closed under single negations.

Proposition 5.3. *If* Σ *is a finite set of formulas, then* $\neg\mathrm{FL}(\Sigma)$ *is finite.*

Definition 5.6 (Atoms)**.** Let Σ be a set of formulas. A set of formulas A is an *atom* over Σ if it is a maximal consistent subset of $\neg\text{FL}(\Sigma)$. That is, A is an atom over Σ if $A \subseteq \neg\text{FL}(\Sigma))$, if A is consistent, and if $A \subsetneq B \subseteq \neg\text{FL}(\Sigma)$, then B is inconsistent. $\text{At}(\Sigma)$ is the set of all atoms over Σ.

Proposition 5.4. $\text{At}(\Sigma) = \{\Gamma \cap \neg\text{FL}(\Sigma) \mid \Gamma \textit{ is maximally consistent}\}$.

Proposition 5.5. *Let Γ be a set of formulas and let $A \in \text{At}(\Sigma)$. Then:*

(i) For all $\varphi \in \neg\text{FL}(\Sigma)$, exactly one of the following of φ of $\sim \varphi$ is in A.

(ii) If $\varphi \vee \psi \in \neg\text{FL}(\Sigma)$, then $\varphi \vee \psi \in A$ if and only if $\varphi \in A$ or $\psi \in A$.

(iii) If $\langle \pi_1 \cup \pi_2 \rangle \varphi \in \neg\text{FL}(\Sigma)$, then $\langle \pi_1 \cup \pi_2 \rangle \varphi \in A$ if and only if $\langle \pi_1 \rangle \varphi \in A$ or $\langle \pi_2 \rangle \varphi \in A$.

(iv) If $\langle \pi_1 ; \pi_2 \rangle \varphi \in \neg\text{FL}(\Sigma)$, then $\langle \pi_1 ; \pi_2 \rangle \varphi \in A$ if and only if $\langle \pi_1 \rangle \langle \pi_2 \rangle \in A$.

(v) If $\langle \pi^* \rangle \varphi \in \neg\text{FL}(\Sigma)$, then $\langle \pi^* \rangle \in A$ if and only if $\langle \pi \rangle \langle \pi^* \rangle \varphi \in A$.

Lemma 5.6 (Analog of Lindenbaum's Lemma)**.** *If $\varphi \in \neg\text{FL}(\Sigma)$ and $\{\varphi\}$ is consistent, then there is an atom $A \in \text{At}(\Sigma)$ such that $\varphi \in A$.*

We are now ready to attempt a first definition of the canonical model.

Definition 5.7 (Canonical model of Σ)**.** Let Σ be a finite set of formulas. The *canonical model over* Σ is the model $\mathfrak{M}^c = (\text{At}(\Sigma), \{S_\pi^\Sigma\}_{\pi \in \Pi}, V^\Sigma)$, where $A S_\pi^\Sigma B$ if and only if $\{\hat{A} \wedge \langle \pi \rangle \hat{B}\}$ is consistent, with $\hat{A} = \bigwedge_{\varphi \in A} \varphi$ and $A \notin \hat{A}$, and for every $p \in \Phi$, $V^\Sigma(p) = \{A \in \text{At}(\Sigma) \mid p \in A\}$.

The problem with $\mathfrak{M}^c$ as just defined is that it is not regular!

Definition 5.8 (Regular canonical model)**.** Let Σ be a finite set of formulas. The *regular canonical model over* Σ is the model $\mathfrak{R} = (\text{At}(\Sigma), \{R_\pi^\Sigma\}_{\pi \in \Pi}, V^\Sigma)$.

The valuation remains the same as in $\mathfrak{M}^c$, but now, for the relations, we have $R_a^\Sigma = S_a^\Sigma$ for every atomica program a, and for complex programs,

$$R^\Sigma_{\pi_1 \cup \pi_2} = R^\Sigma_{\pi_1} \cup R^\Sigma_{\pi_2}$$
$$R^\Sigma_{\pi_1 ; \pi_2} = R^\Sigma_{\pi_1} \circ R^\Sigma_{\pi_2}$$
$$R^\Sigma_{\pi^*} = (R^\Sigma_\pi)^*$$

Lemma 5.7. *For all $\pi \in \Pi$, $S^\Sigma_{\pi^*} \subseteq (S^\Sigma_\pi)^*$, and $S^\Sigma_\pi \subseteq R^\Sigma_\pi$.*

Lemma 5.8 (Existence Lemma)**.** *Let A be an atom and let $\langle\pi\rangle\psi$ be a formula in $\neg\text{FL}(\Sigma)$. Then, $\langle\pi\rangle\psi \in A$ if and only if there is a B such that $AR_\pi B$ and $\psi \in B$.*

Lemma 5.9 (Truth Lemma)**.** *Let $\mathfrak{R}$ be the regular* **PDL***-model over Σ. For all atoms $A \in \text{At}(\Sigma)$ and all $\varphi \in \neg\text{FL}(\Sigma)$, $\mathfrak{R}, A \Vdash \varphi$ if and only if $\varphi \in A$.*

Completeness now follows immediately. We can show the contrapositive: if $\not\vdash_{\mathbf{PDL}} \varphi$, then $\Sigma = \{\neg\varphi\}$ is consistent, so there is an atom $A \in \text{At}(\Sigma)$ such that $\neg\varphi \in A$. By the Truth Lemma, $\mathfrak{R}, A \Vdash \neg\varphi$, so $\mathfrak{R}, A \nVdash \varphi$ and hence $\mathsf{Reg} \nVdash \varphi$.

Theorem 5.10 (Completeness of **PDL**)**. PDL** *is complete with respect to to the class* Reg *of regular frames:* $\text{Log}(\mathsf{Reg}) \subseteq \mathbf{PDL}$.

Corollary 5.11 (Soundness and completeness of **PDL**)**. PDL** *is sound and complete with respect to to the class* Reg *of regular frames:* $\mathbf{PDL} = \text{Log}(\mathsf{Reg})$.

6 Neighbourhood Semantics

The axioms contained by normal modal logics make sense as long as we stick to Kripke semantics. However, in certain contexts, the modalities in the basic modal language can be interpreted in such a way that axioms like K become controversial. An approach to non-normal modal logics requires first a new semantics. A possible option is that of *neighbourhood semantics*.

Definition 6.1 (Neighbourhood frames and models)**.** A *neighbourhood frame* (or *NBD frame*, for short), is a pair $\mathcal{F} = (W, N)$ where W is a nonempty set and N is a *neighbourhood function*,

$$\begin{aligned} N : W &\to \mathcal{P}(\mathcal{P}(W)) \\ w &\mapsto N(w) \subseteq \mathcal{P}(W) \end{aligned}$$

It is sometimes convenient to treat a neighborhood function $N : W \to \mathcal{P}(\mathcal{P}(W))$ as a relation. More precisely, every neighborhood function N corresponds to a relation $R_N \subseteq W \times \mathcal{P}(W)$ such that for any $w \in W$ and $U \in \mathcal{P}(W)$, wR_NU if and only if $U \in N(w)$.

A *NBD* model $\mathcal{M} = (\mathcal{F}, V)$ is a NDB frame $\mathcal{F}$ together with a valuation function $V : \Phi \to \mathcal{P}(W)$.

Definition 6.2 (Satisfaction and validity under neighbourhood semantics)**.** Given a model $\mathcal{M} = (W, N, V)$ and $w \in W$, we say that a formula φ in the basic modal language is *satisfied* in $\mathcal{M}$ at w, written $\mathcal{M}, w \models \varphi$, according to the following conditions:

$$
\begin{aligned}
&\mathcal{M}, w \not\models \bot && \\
&\mathcal{M}, w \models p && \text{iff } w \in V(p) \\
&\mathcal{M}, w \models \neg\varphi && \text{iff } \mathcal{M} \not\models \varphi \\
&\mathcal{M}, w \models \varphi \vee \psi && \text{iff } \mathcal{M}, w \models \varphi \text{ or } \mathcal{M}, w \models \psi \\
&\mathcal{M}, w \models \Box\varphi && \text{iff } [\![\varphi]\!] \in N(w) \\
&\mathcal{M}, w \models \Diamond\varphi && \text{iff } W \setminus [\![\varphi]\!] \notin N(w)
\end{aligned}
$$

The notion of *NBD frame validity*, $\mathcal{F} \models \varphi$, is the same as with the usual Kripke semantics.

Definition 6.3 (Monotone frame)**.** A NDB frame $\mathcal{F} = (W, N)$ called *monotone* if $N(w)$ is upwards closed for every $w \in W$, i.e. $U \in N(w)$ and $U \subseteq V$, entails $V \in N(w)$.

Although we can apply neighbourhood semantics to the basic modal language, we can define a new language too.

Definition 6.4 (NBD language)**.** The *neighbourhood language* is the modal language described by the following grammar, where $p \in \Phi$:

$$\varphi ::= \bot \mid p \mid \neg\varphi \mid \varphi \vee \varphi \mid \langle\,]\varphi \mid [\,\rangle\varphi$$

The intended semantics for the new modalities are the following:

$$
\begin{aligned}
&\mathcal{M}, w \models \langle\,]\varphi : \text{ there is a } U \in N(w) \text{ s.t. for every } u \in U, \mathcal{M}, u \models \varphi \text{ (i.e. } U \subseteq [\![\varphi]\!]) \\
&\mathcal{M}, w \models [\,\rangle\varphi : \text{ for all } U \in N(w) \text{ there is a } u \in U \text{ s.t. } \mathcal{M}, u \models \varphi \text{ (i.e. } U \cap [\![\varphi]\!] \neq \emptyset)
\end{aligned}
$$

Proposition 6.1. *Let $\mathcal{M} = (W, N, V)$ be a monotone NBD model. Then, for every $w \in W$ and every formula φ,*

$$
\begin{aligned}
&\mathcal{M}, w \models \langle\,]\varphi \textit{ if and only if } \mathcal{M}, w \models \Box\varphi \\
&\mathcal{M}, w \models [\,\rangle\varphi \textit{ if and only if } \mathcal{M}, w \models \Diamond\varphi
\end{aligned}
$$

Definition 6.5 (Monotone bisimulations)**.** Let $\mathcal{M} = (W, N, V)$ and $\mathcal{M}' = (W', N', V')$ be two monotone NBD models. A relation $Z \subseteq W \times W'$ is a *monotone bisimulation* if whenever wZw', the following holds:

Atomic harmony For every $p \in \Phi$, $V(p) = V'(p)$.

Zig For every $U \in N(w)$, there exists $U' \in N'(w')$ such that for all $u' \in U'$, there is a $u \in U$ such that uZu'.

Zag For every $U' \in N'(w')$, there exists $U \in N(w)$ such that for all $u \in U$, there is a $u' \in U'$ such that uZu'.

Whenever two points $w \in W$, $w' \in W'$ are *bisimilar*, we write $\mathcal{M}, w \leftrightarrows \mathcal{M}', w'$.

Theorem 6.2 (Monotonic Bisimulation Theorem)**.** *Let $\mathcal{M}$ and $\mathcal{M}'$ be NBD models such that $\mathcal{M}, w \leftrightarrows \mathcal{M}', w'$, and let φ be a formula in the NBD language. Then, $\mathcal{M}, w \models \varphi$ if and only if $\mathcal{M}', w' \models \varphi$.*

In a way, neighbourhood frames can be seen as an extension of Kripke-frames. The following two theorems show how we can translate between domains.

Theorem 6.3 (Kripke frames as monotone NBD frames)**.** *Let $\mathfrak{F} = (W, R)$ be a Kripke frame. Then, the neighbourhood frame $\mathcal{F} = (W, N_R)$, where for every $w \in W$ the neighbourhood function is $N(w) = \{U \subseteq W \mid R[w] \subseteq U\}$, satisfies that for every formula φ, every valuation V and every $w \in W$,*

$$\mathfrak{F}, w \Vdash \varphi \text{ if and only if } \mathcal{F}, w \models \varphi$$

Similarly, we can see a monotone neighbourhood frames as a Kripke frames. But, in fact, we impose some extra condition:

Definition 6.6 (Augmented NBD frame)**.** A monotone NBD frame $\mathfrak{F} = (W, N)$ is called *augmented* if

(i) If $U, V \in N(w)$, then $U \cap V \in N(w)$.

(ii) $W \in N(w)$

(iii) $\bigcap N(w) \in N(w)$

Theorem 6.4 (Augmented NBD frames as Kripke frames)**.** *Let $\mathcal{F} = (W, N)$ be an augmented NBD frame. Then, the Kripke frame $\mathfrak{F} = (W, R_n)$, where for every $w, v \in W$, wR_Nv if and only if $v \in \bigcap N(w)$, satisfies that for every formula φ, every valuation V and every $w \in W$,*

$$\mathcal{F}, w \models \varphi \text{ if and only if } \mathfrak{F}, w \Vdash \varphi$$

Definition 6.7 (Neighbourhood logics)**.** The logic **E** is the smallest set of formulas containing all propositional tatutologies, the axiom (Dual) and closed under the rules (MP), (US) and

$$\frac{p \leftrightarrow q}{\Box p \leftrightarrow \Box q}\ \text{(RE)}$$

Besides, we can consider axioms

$$\Box(p \land q) \to \Box p \land \Box q \tag{M}$$

$$\Box p \land \Box q \to \Box(p \land q) \tag{C}$$

$$\Box\top \tag{N}$$

and use them to define the logics **EM** = **E**+M, **EC** = **E** + C, **EMC** = **EM** + C and **EMCN** = **EMC** + N. (This last one is precisely **K**: **EMCN** = **K**, as (N) can be used to simulate the (Nec) rule.)

6.1 Soundness and completeness for E and EM

We give two completeness results for NBD logics: that **E** is sound and complete with respect to the class of all NBD axioms; and that **EM** is sound and complete with respect to the class of all monotone NBD axioms.

Soundness follows immediately. The main step is proving that (RE) preserves validity under any NBD frame.

Theorem 6.5 (Soundness of **E**)**.** *The logic* **E** *is sound with respect to the class of all neighbouhood frames.*

For completeness, we use, as usual, the canonical model construction. We take for granted the usual definitions regarding consistency, maximally consistent sets, and so on.

Definition 6.8 (Canonical NBD model)**.** We define $\mathcal{M}^c = (W^c, N^c, V^c)$, the *canonical NBD model*, as

- $W^c = \{\Gamma \mid \Gamma \text{ is a maximally consistent set}\}$
- $N^c(\Gamma) = \{|\varphi|_{\mathbf{E}} \mid \Box\varphi \in \Gamma\}$, where $|\varphi|_{\mathbf{E}} = \{\Gamma \in W^c \mid \varphi \in \Gamma\}$ (sometimes called the *proof set* of φ)
- $V^c(p) = \{\Gamma \in W^c \mid p \in \Gamma\}$

With some intermediate work, we can prove a Truth Lemma.

Lemma 6.6 (Truth Lemma)**.** *For every formula* φ*,* $[\![\varphi]\!] = |\varphi|_{\mathbf{E}}$*. That is, for every maximally consistent set* Γ*,* $\Gamma \models \varphi$ *if and only if* $\varphi \in \Gamma$*.*

Completeness follows immediately: if $\nvdash_{\mathbf{E}} \varphi$, then $\{\neg\varphi\}$ is **E**-consistent, so there exists a maximally consistent set Γ such tat $\neg\varphi \in \Gamma$. Hence, $\varphi \notin \Gamma$, and by the Truth Lemma, $\Gamma \not\models \varphi$, so $\not\models \varphi$.

Theorem 6.7 (Completeness of **E**)**.** *The logic* **E** *is complete with respect to the class of all neighbourhood frames.*

An analogous result can be proven for **EM** with respect to the class of all monotone neighbourhgood logic. Note that it that case, the canonical model is not directly monotone, so we need to close it upwards and reprove the Truth Lemma and the intermediate work to prove completeness.

Theorem 6.8 (Soundness and completeness of **EM**)**.** *The logic* **EM** *is sound and complete with respect to the class of monotone neighbourhood frames.*

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