Between Invention and Discovery: What Video Game Speedrunning Can Teach Us about Mathematical Practice

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February 8, 2022

Abstract

Speedrunning is a widespread activity in which players try to complete a video game as fast as possible. It requires both skill at controlling the game and, more importantly, a good understanding of how the game works and what glitches and tricks the game's code contains. These are researched by a community of glitch hunters, who try to understand how the game works to find ways around the main obstacles, always without making the program crash.

Mathematical practice and speedrunning resemble each other in how they both revolve around objects determined by rigid human-made rules that can nevertheless lead to unexpected discovery and invention. This paper sets up such an analogy between both practices to support the constraint-based account of mathematical practice proposed by authors such as Wagner (2017), according to which mathematics is a human activity constrained by social and cognitive constraints that shape (and are reshaped by) the practice. The analogy is explored both at a general level and through particular case studies based on episodes of the history of mathematics and speedrunning.

1 Introduction

The philosophy of mathematics is stuck in endless scholarly debates, yet its most pressing issue boils down to a simple question: is math invented or discovered? If it is invented, then why does it explain nature so perfectly? And why do we play by its strict rules and methods? If it is all made up, why is not everything allowed in mathematical practice? It must then be because mathematics is not made up but very much beyond human construction, in a perfect God-given Platonic realm! But then, how do we access this realm? How do we account for the ambiguous, confusing and mistaken mathematics we produce? How do we argue for the mathematics modelling phenomena humans created from scratch?

In short, is mathematics dominated by natural order or by conceptual freedom? Despite significant efforts from philosophers during the last century, no consensus has been reached. The mathematical community might have converged towards a formalist–structuralist–Platonist mindset[1], but the questions of whether math and its rules are invented or naturally God-given remains fundamentally unresolved.

Understandably so, since what the philosophy of mathematics is trying to settle is a false dichotomy: mathematics is both invented and discovered, both ordered and free. Mathematics is a constrained practice; it is "invented" in that it is something made by humans, from scratch, with no Platonic creator of finished mathematical truths. However, that does not mean that anything goes. Very much on the contrary: mathematics is constrained. It is a creative process carried out by humans, who invent ideas, objects, and concepts, subject to their cognitive abilities, existing social constraints, values, prevailing knowledge transfer structures, institutional limitations, and much more. Around these rules and constraints, different processes of reinterpretation happen. Some of them involve "discovery" regarding what some of these more or less invisible rules (dis)allow; others require novel reinterpretation and creative endeavour, and often it is both. Stating a theorem does not make it true to the eyes of the community. Instead, the theorem must be possible inside the system of rules and constraints that we operate within. To do mathematics is to navigate this web of visible and invisible constraints.

Though this description could apply to most if not all human practices, mathematics is somewhat unique in the rigidity of the rules employed and the seeming universality of its theorems and results. There is, however, another practice that resembles mathematics in the way it handles rules, invention and discovery around them: speedrunning in video games.

This paper draws an analogy between mathematics and speedrunning, supporting the view of mathematics as a constrained cultural practice. It tries to escape the existing preconceived ideas developed by philosophers and mathematicians. It intends to show that what is natural in a different setting can also be correct and reasonable in ours.

The structure of the paper is as follows. The next two subsections introduce speedrunning and briefly outline the so-called constraint-based approach to the philosophy of mathematical practice. Section 2 presents a general analogy comparing mathematics to speedrunning, building on the work of game scholars such as Hemmingsen (2021) and the conceptual framework of metagames of Boluk and LeMieux (2017). Section 3 zooms in deeper and establishes local analogies between specific episodes of mathematics and speedrunning, further supporting the thesis of mathematics seen as a cultural practice built around the manipulation of contingent rules.

1.1 Speedrunning: the game must be fast

In the world of video games, a speedrun is a play-through where a player tries to complete the game as fast as possible. Though what it means to "complete a game", as well as what "fast" means, may vary from game to game, one often thinks of it as reaching the end credits of the game in as little time as possible. As the much-debated definition by Skully-Blaker puts it, it is "the practice of players or 'runners' attempting to 'travel' from a game's opening state at its

1. Formalist in a strict sense, because it relies on the existence of formal proofs as the ultimate mechanism for arbitration, and in a loose sense, because it follows Bourbaki's style of rigour via formalization in natural language; structuralist, because mathematical logic has shown that there is nothing canonical about any specific foundational framework; and Platonist, because of the universality attributed to those structures.

first necessary button input to the game's conclusion at its last necessary button input in the smallest amount of time possible" (2014). Speedrunning is practised by thousands of players worldwide, who share their runs by streaming them live and keeping record leaderboards, such as those on www.speedrun.com.

The most exciting aspect of speedrunning is often the inventiveness used to complete the games so quickly. For example, a game like *The Legend of Zelda: Ocarina of Time* (Nintendo, 1998), one of the most popular titles in the speedrunning community, is completed by the best runners in just over six minutes, while a typical player would usually need around 25 hours of gameplay. How is this possible? Speedrunners use glitches and tricks to clip through walls, distort the game world's physics, gather items earlier than they otherwise would, or even skip entire sections of the game. Any glitch which does not crash the game is considered fair play. In order words, virtually anything that is already in the game's code is considered a legal move, even if (or rather precisely because) it goes against the intention of the game designers.

Alongside the speedrunners, there are the so-called glitch hunters. These are players devoted to finding and documenting glitches, trying to understand how the game works, and reverse-engineering the game code. Glitch hunters and speedrunners work together worldwide, sharing discoveries that are then combined to invent intricate tricks that open new routes in the games. Finally, highly-skilled players put to use all the tricks developed by the community, skipping sequences and getting to the end in an attempt to break the existing records.

A simple analogy already emerges between mathematics and speedrunning. As in mathematics, speedrunning involves a community of people researching an object (a game) with strict rules (provided by the game's code) where anything is allowed as long as it does not break the system. The practice involves a high degree of mechanical skill, luck, patience, and insight, which together can lead to discoveries and a great deal of invention.

The status of speedrunning and its authorial, political and aesthetic implications for the video game medium have been the topic of recent research and debate in game studies. This paper builds on the work of authors involved in these debates, such as Scully-Blaker (2016), Boluk and LeMieux (2017), Hemmingsen (2021) or Ricksand (2021).

1.2 A constraint-based account of mathematical practice

Mathematics is often regarded as an edifice too perfect to be simply the product of human invention. However, video games are apparent cultural objects created from scratch by human beings, conditioned by social and cognitive constraints. By drawing an analogy with video games, one could transport some of the open questions in the philosophy of mathematics and mathematical practice to a more specific context where the activity is still similar.

Crucially, this is an analogy that works at the level of practices. We are interested in what mathematicians and speedrunners *do* and *why* they do it. The key insight here is that mathematics is built around the imposition of human-made rules. After all, mathematics is the product of "internal" and "external" pressures, cognitive and sociological. As such, understanding mathematics means understanding how its practice changes, evolves and adapts; how groups, institutions and systems of values shape the practice back and forth, determining what mathematicians both invent and discover. In short, mathematics is a cultural system. And as such, natural order and conceptual freedom are just two faces of the same coin. As Wilder writes,

> there is the belief expressed by G.H. Hardy that "mathematical reality lies outside us, and that our function is to discover or observe it, and that the theorems which we prove, and which we describe grandiloquently as our 'creations' are simply our notes of our observations." On the other hand, there is the point of view expressed by P.W. Bridgman that "it is the merest truism, evident at once to unsophisticated observation, that mathematics is a human invention." Although these statements seem irreconcilable, such is not the case when they are suitably interpreted. For insofar as our mathematics is a part of our culture, it is, as Hardy says, "outside us." And insofar as a culture cannot exist except as the product of human minds, mathematics is, as Bridgman states, a "human invention." (Wilder 1998, 188)

A more fledged-out framework for the philosophy of mathematical practice that turns useful to study this analogy is given by Wagner (2017). He identifies mathematical practice as being fundamentally constraint-based. His theoretical framework explains mathematical production, postulating that mathematics is a human practice conditioned by social and cognitive constraints. As such, mathematical discovery and invention are contingent processes of reinterpretation around the semiotic systems used in the practice.

Wagner's work has illustrated this claim through a variety of historical episodes. In this paper, we approach a defence of these philosophical claims through a different lens. We translate between practices to see that what seems natural in one can also be natural in the other, and what is regarded as nonsense in one is probably also nonsense in the other.

2 A macroscopic analogy

Video game speedrunning and mathematics are analogous at three different levels, all essential to both practices: they share (*i*) a fixed object of study built around hardcoded rules that cannot be violated (i.e., a video game's code / an axiomatic system), (*ii*) a looser set of soft rules that can be partially violated, modified or broken (a set of socio-cognitive constraints) and (*iii*) a community of people who study the objects in question to get new collective insights (the speedrunning community's glitch hunting / the mathematical community's research).

Since I assume the reader to be familiar with contemporary mathematical practice, let us say something about speedrunning. Each speedrunning community is built around a specific video game. Some of the most popular titles are classic games such as *Super Mario 64* (Nintendo, 1996) or *Ocarina of Time*, but virtually any video game can be the subject of speedrunning, and in fact most are.

A particular speedrunning community revolves around two factors. On the one hand, the video game in question, whose rules are implicitly written in the game's code. For *Super Mario 64*, these rules determine how high Mario can jump, what walls can be traversed and which cannot, what levels must be completed and in what order, and so on. On the other hand, there are rules regarding the speedrun type and the players' physical abilities. For most games, three main categories exist: Any% (the most popular, where players try to reach the end credits as fast as possible), 100% (where players need to complete all quests, side missions and gather all collectable items on top of finishing the game) and Low% (the opposite of 100%, trying to complete the game with as few objects as possible)[2].

Speedrunning is the paradigmatic example of a metagame, as defined by Boluk and LeMieux (2017): a game about, around and on top of a game. Arguably, the notion of metagame is quite broad. To a certain extent, everything is a metagame. Boluk and LeMieux use this framework to comprise a wide range of practices built around and on top of video games: streaming of *let's plays*, mods, educational tools, indie games[3], and more. Not only can so many practices be seen as metagames, but we are not aware that they are for the most part. As they put it, "the greatest trick of the videogame industry ever pulled was convincing the world that videogames are games in the first place" (Boluk and LeMieux 2017, 8). Ultimately, "humans make their own games, but they do not know about it" (5).

Metagames are determined by two types of constraints, which Boluk and LeMieux call the *mechanics* and the *rules*. Mechanics are more or less fixed and rigid. In the case of speedrunning, this is the game's code. The rules, on the other hand, are what make the practice meaningful. In speedrunning, rules are, amongst other things, the category, whether the run is glitchless or not, what input methods are allowed or whether specific exploits are permitted. In this way, the same mechanics can lead to different metagames by supplying different sets of rules on top.

We thus have the first two elements of the analogy between speedrunning and mathematics: the set of mechanics and the set of rules. For speedrunners, the object of interest imposing the mechanics is a given video game, while for the mathematician, this is either an axiomatic system like ZFC or, in a looser sense, some conceptual environment, consisting of a battery of definitions and ideas, such as number theory, topology or some other branch of mathematics. The rules imposed by the mechanics are rigid because breaking them is simply not possible. The player cannot make Mario jump higher than the game's code allows. Similarly, the mathematician cannot make 24 be a prime number without violating the definitions.

As Hemmingsen (2021) puts it, "Code is Law". Speedrunners think of code as natural law, determining the possibilities of the game world in the same way that physical law determines the possibilities of our physical reality. Observe, however, that despite the mechanics being understood as natural law, they are very much artificial. After all, the game is the product of designers and programmers, who wrote the code from scratch and accidentally introduced bugs that the players exploit. Though unbreakable, there is nothing Platonic about this first layer of constraints.

2. For example, when playing in the Low% category, a player of *Ocarina of Time* who completes the game in one hour by picking only the Kokiri Sword wins against a player who picks both the Kokiri Sword and the Master Sword and completes the game in 30 minutes —while the latter player would win in the Any% category. None of those runs would be valid in the 100% category, though, since neither player collected the optional Biggoron's Sword.

3. By "indie game", Boluk and LeMieux refer to the specific movement stemming in the late 2000s and early 2010s from *Braid* (Jonathan Blow, 2008), *Super Meat Boy* (Edmund McMillen and Tommy Refenes, 2010) and *Fez* (Polytron, 2012), and especially as depicted in the documentary *Indie Game: The Movie* (James Swirsky and Lisanne Pajot, 2012). They are metagames because they are games *about* the classic platforming games of the 80s.

The second layer of constraints determines the metagame. In speedrunning, this is the category and the player's physical abilities. In mathematics, these are, for example, the proof techniques allowed, but also the broader social and cognitive constraints under which the mathematician operates: the trends that determine what topics are interesting and publishable, the time constraints under which research has to be carried out, and much, much more (see (Wagner 2017, ch. 3) for a more detailed discussion on the constraints of mathematics).

The critical insight is finally brought by the third element of the analogy: the community and its activity. The analogy is not that doing mathematics is like performing a perfect speedrun of *Super Mario Bros.* (Nintendo, 1985), but rather that doing mathematical research is like doing research for speedrunning. Researching a game for speedrunning purposes is often called *glitch hunting*. In glitch hunting, players work together to try and understand the intricacies of the game's code. After all, code may be law, but laws have loopholes. For example, in most circumstances in *Ocarina of Time*, the main character, Link, can jump a fixed distance, determined by the game's code. This distance is short enough so that Link can only reach the areas the game designers intended. However, if he places a bomb next to him and then jumps while the bomb is exploding, the expansive wave will make him travel a longer distance, reaching places otherwise inaccessible. Observe that this is all allowed by the game code. However, that this chain of events is possible is far from obvious. The typical player of *Ocarina of Time* will complete the game without ever noticing this. Practice and research are required to find this trick.

For glitch hunters, however, the previous technique would likely not count as a proper glitch. Sure, performing it seems to go against the designers' intentions, but they nevertheless contemplated how Link is supposed to react when hit by an explosion, and they programmed it into the game; they simply did not expect the use of the feature to access locked areas early. A glitch is something the programmers left unprogrammed. It is either a bug or some limit case they forgot to specify. For example, in *The Legend of Zelda: Breath of the Wild* (Nintendo, 2017), the player can make Link fly[4] by performing a particular sequence of moves (Linkus7 2021). This button input is so involved that the typical player will not perform it even by accident, and likely, the programmers at Nintendo never thought about it.

The community of glitch hunters shares the tricks, glitches and bugs they find, showcasing them and contributing to a partial knowledge of how the game works. These glitches are later combined in inventive ways that lead to proper strategies that skilled players perform to break world records. Arguably, these glitches are very much discovered. Neither the players nor the programmers knew of their existence. And yet, the strategies are invented: though glitch hunters discovered that Link could fly, this technique does not provide per se a way to complete the game faster. It is only when players inventively combine different glitches and techniques that they develop a new "route" to beat the game faster. It is all a combination of accidental discovery and meticulous crafting.

In short, glitch hunting and mathematics have a lot in common. Ultimately, they are both activities amidst invention and discovery around rules that human agents created. This macro-

4. Though *Breath of the Wild* takes place in the fantasy kingdom of Hyrule, and Link has some magic powers available to him, flying is not one of them, and in fact, this contradicts the narrative of the game. A discussion of how narrative elements and the philosophy of fiction can be used to decide on what is considered a glitch and what an intended feature has been provided by Ricksand (2021).

scopic analogy goes even further, as it also partially extends to the values that the respective communities endorse. That is, the ethos of speedrunning and mathematics are pretty close.

Hemmingsen posits the ethos of speedrunning to comprise three characteristics: "constitutive skills (including dexterity, memorisation and mental fortitude); a collective, fine-grained knowledge of the game; and the desire to subvert the intentions of the programmers" (2021, 435). Moreover, there is an order of relevance between these: "collective knowledge takes priority over constitutive skills, and subversion takes priority over both" (2021, 435). I come back to subversion in speedrunning and mathematics in Section 3.3. For now, I focus on individual skills and collective knowledge. Indeed, the best mathematicians are often people with remarkable constitutive skill. However, regardless of the virtuoso aptitude,

> the mathematical genius can only carry on from the point which mathematical knowledge within his culture has already reached. Thus if Einstein had been born into a primitive tribe which was unable to count beyond three, life-long application to mathematics probably would not have carried him beyond the development of a decimal system based on fingers and toes. (Linton 1936, 319)

In the same way, the most dexterous player using the Nintendo 64 controller will be lost without the existing extensive collective knowledge of *Ocarina of Time*, developed by hundreds of players all around the world for more than two decades. The collective aspect of knowledge is vital: "As a body of knowledge, mathematics is not something I know, you know, or any individual knows: It is a part of our culture, our collective possession. I may even forget, with the passing of time, some of our own individual contributions to it, but these may remain, despite our forgetfulness, in the culture stream" (Wilder 1998, 188). In this way, we may forget who was the first player to complete *Ocarina of Time* under 15 minutes, but the technique and the knowledge will remain in all subsequent runs by other players.

Ultimately, if human beings create video games that allow for unexpected discovery and invention, why should mathematics be any different?

3 Microscopic analogies

The analogy presented above is macroscopic in that it only works on a general level. One may ask whether the body of mathematics is equivalent to one specific game or whether perhaps each game relates to some area. That would be taking the analogy too far, since the different branches of mathematics interact between them, while games are isolated objects for the most part[5]. More importantly, the analogy relies on a clear-cut distinction between the mechanics and the rules of each practice. However, as soon as we look closer into both practices, we note that the line separating the mechanics from the rules is blurry and less law-like than it initially appears. A grey area looms between unbreakable mechanics and conventional rules.

5. In some cases, knowledge of one game can translate to another, but this is rare. For example, various *The Legend of Zelda* or *Grand Theft Auto* titles are programmed using the same graphics engine, meaning specific bugs and glitches are transferred between games. This, however, is an exception.

Crucially, both practices work around a continuous interpretation of the mechanics, constantly redrawing the boundaries of what is allowed. For example, in speedrunning, hardware differences can make the code behave differently, making it less law-like. In mathematics, the mechanics are constantly refined to decide whether the objects of interest fall under the existing definitions (consider, for example, Lakatos' (1963) discussion of the possible definitions of polyhedra in the context of Euler's characteristic formula). Even the seemingly hardest rule is always somehow under-determined when it comes to deciding how it would be applied.

Studying this grey area can help draw a more nuanced portrait of both practices, understanding how the changing boundaries are renegotiated and under what criteria. To this end, I now turn the attention to particular case studies in which speedrunning can shed light on mathematical practice. I start tackling the issue of subversion, which Hemmingsen considers the primary motivation of speedrunners, and how self-reference in Hilbert's programme can be seen as a glitch. I then turn my attention to the race between speedrunners for better times, comparing the research for faster multiplication algorithms with the quest for capture-free runs in *Super Mario: Odyssey* (Nintendo, 2017). Finally, the last subsection asks what criteria are used to choose between rules and categories and what tools are allowed in speedrunning compared to mathematics, looking into the grey area looming around the division between mechanics and rules.

3.1 Subversion and glitch hunting in Hilbert's programme

For Hemmingsen, the ethos of speedrunning comprises constitutive skill, collective knowledge of the game and subversion against the game designers. He claims that the latter is the most important of all three attributes in that it is what speedrunners value most. He argues that it is only by seeing subversion as the ultimate goal of speedrunning that one can understand, for example, why cheat codes are not allowed by the community[6].

Subversion is often not that clear in mathematics, mainly because there is no specific designer. One could consider the designer the mathematician(s) who came up with some definition or conceptual framework. However, as time goes by, such a framework will be assimilated by the ahistorical and impersonal voice predominant in mainstream mathematical discourse. Authorship will be diluted.

One can then imagine two types of mathematicians. On the one hand, those who play by the rules and try to connect different branches of mathematics, solve open problems, and prove conjectures, all in an endeavour reminiscent of a Kuhnian state of normal science. On the other hand, there are people who rebel against the existing methods and tradition, such as Georg Cantor or John Wallis, who prided themselves in going against the existing rules. In speedrunning, the former, most abundant type of mathematician would be what Skully-Blaker calls a "finesse runner", while the people looking for more spectacular runs based on glitches are "deconstructing runners":

> The distinction between finesse and deconstructing speedruns relates to the assumption that games are narrative spaces, something that simply means that games de-

6. The issue of cheat codes is discussed in Section 3.3, under the context of rule choice.

> liver stories of varying depth through their virtual worlds. In this context, finesse runs are those in which the narrative architecture of the gamespace is largely left intact — a finesse run looks like a very efficient version of a normal playthrough of a game. A deconstructing run, on the other hand, is one in which narrative boundaries are torn down by the player through the use of glitches and other programming oversights. (Scully-Blaker 2016, 4)

Let us look at two paradigmatic examples of deconstructing runs in mathematics: the justification of Cantor's transfinite numbers in the axiomatic setting of modern set theory and self-reference in Gödel's incompleteness theorem.

For Cantor, the fact that one could keep counting beyond the natural numbers was quite intuitive, his initial view of the well-ordering theorem being that it was an obvious "law of thought". Under a contemporary axiomatic reconstruction of set theory, however, the introduction of transfinite ordinals feels like a disruptive move. If one defines zero to be the empty set and the successor operation à la von Neumann to be the function $s(x) = x \cup \{x\}$ (or $s(x) = \{x\}$, à la Zermelo) then nothing prevents you from applying s beyond the sets taken to be natural numbers. It is in this glitchy way that one obtains $s(\mathbb{N}) = \omega + 1$ and some of the subsequent transfinite ordinals. It is perfectly allowed, yet for somebody new to transfinite arithmetic, this application of the set-theoretic successor function feels uncomfortable.

If the application of the successor function to any set is something along the lines of bomb-jumping in *Ocarina of Time* (a trick), then self-reference in formal systems is a proper glitch. Under David Hilbert's finitist programme, one was to reduce all mathematics to symbol manipulation in some object language, a formal system such as the first-order predicate calculus. If one could formalize mathematics in such a system and prove its consistency at a metamathematical level, then the epistemological basis of mathematics would be secured by the certainty provided by Kantian intuition over finite objects.

In 1929, Kurt Gödel made a vital contribution to Hilbert's programme when proving that the predicate calculus was complete with respect to the existing proof systems: every sentence semantically entailed by a set of axioms can also be formally derived from those axioms. However, in 1931, his two celebrated incompleteness theorems would partially put an end to Hilbert's dream. In its contemporary formulation, the incompleteness theorems rely on the Gödel sentence, a formula in the first-order language of arithmetic claiming its own unprovability from the axioms. The key insight was devastating: a formal system strong enough to express elementary arithmetic was powerful enough to talk about itself in some sense.

Self-reference made the axiomatic systems Hilbert envisioned as the foundation of mathematics incomplete. The fact that a formal system could talk about itself was surely not intended, making its discovery a proper glitch: it is allowed by the code to the point that it can be developed into a powerful feature, but the designers do not intend it. Under this perspective, Gödel was a glitch hunter who discovered the glitch of self-reference.

However, anyone familiar with the theorems knows that the proofs are far from trivial. It is more accurate to say that Gödel discovered not self-reference but rather a variety of glitches he exploited to invent a strategy leading to the final Gödel sentence. In contemporary proofs, one first needs to provide a suitable encoding of formulas into numbers. One should then ensure that primitive recursion schemes are expressible under this encoding and that all the necessary predicates to express provability are expressible in the language of arithmetic. Even then, the proof of the Diagonal Lemma yielding the Gödel sentence is far from trivial. Again, Gödel was dealing here with both discovery and invention. In glitch hunting terms, these would have been various glitches and exploits combined together over months or years of work.

Furthermore, subsequent work in mathematical logic resembles the way strategies are optimized by runners. In Gödel's proof, the axiomatic system being proven incomplete is required to be ω-consistent, with this requirement later dropped by the so-called Rosser trick, due to J. Barkley Rosser in 1936. In glitch hunting, it is often the case that runners improve and perfect routes by playing through the game, making techniques easier to perform as time goes by[7]. Like mathematics, glitch hunting is an iterative process in which existing knowledge is combined and refined with discoveries, leading to new ways to approach the game.

3.2 Speedrunning multiplication: from Strassen's algorithm to capture-free runs in *Super Mario: Odyssey*

In 1969, German mathematician Volker Strassen set out to prove that the naive algorithm for matrix multiplication is essentially optimal. The problem is simple: given two $n \times n$ matrices over some field, compute their product.

How fast can this be done? The naive algorithm immediately follows from the definition of matrix products: compute the inner product of every row of the first matrix by every row of the second matrix. For two 3×3 matrices, this takes 8 multiplications. In general, the algorithm has cubic running time, requiring $O(n^3)$ multiplications.

At first glance, there seems to be no better way of completing the task. A more efficient method would require departing from the very definition of matrix multiplication. However, in 1969, Strassen surprised the community and himself with a smart technique that could slightly reduce the number of operations. As Robinson (2005) points out, the idea was reminiscent of Gauss, who had noted that the product of two binomials $(a + b)(c + d) = ac + ad + bc + bd$ requires four multiplications in the standard approach but can be reduced to three by noting that $ad + bc = (a + b)(c + d) - ac - bd$, at the cost of a few extra additions and substractions (which happen to be cheaper than multiplications). Strassen managed to apply a similar idea to a block decomposition of square matrices that allowed to solve matrix multiplication in time $O(n^{\log_2 7})$, approximately $O(n^{2.8074})$.

His work propelled fruitful research leading to ever-faster algorithms for matrix multiplication. Subsequent improvements and the invention of the so-called "laser method" led to a $O(n^{2.3755})$ bound in 1987, due to Coppersmith and Winograd (1987). Recent sophisticated variants of this method have brought down the asymptotic complexity even further, though only by small factors, with the current record being due to Alam and Williams (2021), with time $O(n^{2.3728596})$ (see Table 1 for a summary of the improvements).

7. For example, the glitch in *Breath of the Wild* that lets Link fly used to rely on equipping some specific objects. It was then noted that bombs suffice, and Link has an infinite supply of those available to him at any time, making the glitch easier to execute in situations where the player has fewer objects with them.

Year	Bound on ω	Authors
1969	2.8074	Strassen (1969)
1978	2.796	Pan (1978)
1979	2.780	Bini, Capovani and Romani (1979)
1981	2.522	Schönage (1981)
1982	2.517	Romani (1982)
1982	2.496	Coppersmith and Winograd (1982)
1986	2.479	Strassen (1986)
1987	2.3755	Coppersmith and Winograd (1987)
2010	2.3737	Stothers (2010)
2012	2.3729	Williams (2012)
2014	2.3728639	Le Gall (2014)
2021	2.3728596	Alman and Williams (2021)

Table 1: Progress on the matrix multiplication exponent, as of February of 2022.

The immediate question is: how far can this be taken? Not too far, since matrix multiplication requires at least n^2 operations, simply because the resulting matrix has n^2 entries to be written. Theoretical computer scientists denote by ω the *matrix multiplication exponent*: the least real number such that there is an algorithm computing matrix multiplication in $n^{\omega+o(1)}$ operations. Today, based on the naive lower bound and the best-known algorithm, we know that $2 \leq \omega \leq 2.3728596$, with some researchers conjecturing that $\omega = 2$ will eventually be achieved[8].

In October of 2017, Nintendo released *Super Mario: Odyssey*, a return to exploration-based 3D Mario games. The title includes a new mechanic: Mario's hat (Cappy) can be tossed, with the ability to "capture" objects and enemies. In this way, for example, Mario can throw Cappy against a goomba (the most common enemy in the game) to capture it, letting the player take control of the goomba. The game is fundamentally built around this mechanic, forcing the player to use it to complete multiple sections of the game.

The speedrunning community around *Super Mario: Odyssey* soon created a new category called Minimum Captures, aiming at completing the game with as few captures as possible[9]. Only a month after release, players could complete the game with 15 captures only, most of them seemingly impossible to avoid. In May of 2018, modders found out that the final capture of the game is technically part of a cutscene, so it could be skipped, bringing the count down to 14. By October of that year, a technique known as "nut jumping"[10] allowed players to avoid an extra capture, making it to 13. Interestingly, the glitch was discovered in November of 2017, only a month after release, but it was not properly incorporated into a run until October 2018. By May

8. Not Strassen, though: "Vassilevska Williams remembers a conversation she once had with Strassen about this: 'I asked him if he thinks you can get [exponent 2] for matrix multiplication and he said, 'no, no, no, no, no.''" (Hartnett 2021)

9. Lowest Percent has covered the development of the category up to 2020 in a short video available on YouTube (Lowest Percent 2020a). Most of the material from this section comes from there.

10. The technique involves the player repeatedly running and hitting a nut that has touched water to make Mario reach arbitrary height. The move must be executed in time windows of two frames (since the game runs at 60 frames per second, that means the player has $\frac{1}{30}$-th of a second to press the button), making it technically very demanding.

Date	Captures (κ)	Achievement
Oct. 2017	-	Game launches.
Nov. 2017	16	-
Nov. 2017	15	Chain-Chomp skip in Cascade Kingdom.
May 2018	14	Final capture turns out to be a cutscene.
Oct. 2018	13	Nut jumping to skip the last wire in Bowser's Kingdom.
May 2019	10	Boss fight avoided in Cascade Kingdom.
Feb. 2020	9	Pokio skip in Bowser's Kingdom.
Feb. 2020	3	Cappy Return Cancel (CRC) discovered and executed.

Table 2: Progress on *Super Mario: Odyssey*'s Minimum Captures category, as of February 2022.

of 2019, several discoveries were made that led to a new route with ten captures only. This route involved playing with two controllers in the Two-Player mode, where one plays as Mario and the other controls Cappy. Since finding another player skilled enough to speedrun in pairs is extremely difficult, the prominent names in the community learned how to use the second controller with their feet, which is now a standard in the Minimum Captures category. Finally, by February of 2020, an advanced frame-perfect technique known as Cappy Return Cancel (CRC) was developed. If properly executed (often requiring a metronome for time accuracy) and in the correct setup, the glitch allows Mario to teleport, getting the runs down to just three captures, where the world record remains to this date (Table 2 summarizes the main milestones in the Minimum Captures Category).

Analogously to the matrix multiplication exponent, we could denote by κ the minimum natural number such that *Super Mario: Odyssey* can be completed with κ captures. Today, we know that $\kappa \leq 3$, with some players claiming it will eventually reach $\kappa = 0$, what would be a *capture-free run*. Ultimately, whether a capture-free run is possible (whether it "exists", if one is inclined to adopt a more mathematical ontology) depends on a wide variety of factors: indeed, the game's code, but also our understanding of it, the invention of new tools and gadgets to interact with the game, the discovery of new glitches, new hardware modifications and whatever else comes between a code and the actual playing of the game.

The research on matrix multiplication algorithms and capture-free runs is similar not just in form but also in purpose, or rather in the seeming lack thereof. Matrix multiplication is a standard algorithm used in many linear algebra and graph theory applications, the complexity of such multiplications often becoming the main bottleneck of the rest of the procedure. It might seem that finding faster algorithms for matrix multiplication has an immediate practical application, but this is actually not really the case. Though the asymptotic complexity is lowered, the constants hidden by the Big-O notation are so astronomically large that the usual naive algorithms perform faster and are far simpler to implement for most purposes (Robinson 2005). The quest for faster algorithms is thus purely intra-theoretical: it cannot be justified by some critical applications in industry but rather by the genuine interest of researchers. This is part of a process of "dismotivation", a phenomenon identified by Wagner as the "gradual loss of a mathematical statement's empirical motivations and grounding" (Wagner 2017, 60). For him, dismotivation is one of the constraints that makes something be considered mathematics proper. After successful dismotivation, empirical grounding is lost, and research questions arise intra-theoretically or diegetically.

Furthermore, both examples highlight the vague boundary between the mechanics and the rules. In *Super Mario: Odyssey*, players are allowed to use a second controller with their feet as well as a metronome for time accuracy, but they are not allowed, for example, to run the game into a modified system. It is all about interpretation. Similarly, computer scientists are allowed to move past the original matrix multiplication definition, pushing new interpretations.

So will something happen if $\omega = 2$ is ever proven? In practical terms, no, but the proof will likely introduce new ideas useful in future research in arithmetic complexity. Ultimately, the same happens with *Super Mario: Odyssey* speedruns. Players make their lives more difficult by playing under these rules. They distort the game's careful harmony, making it impossible to enjoy its narrative, intended flow and level design. Crucially, both advanced matrix multiplication algorithms and low-capture runs are disrupting and subversive in that they rebel against the designer's intentions. This is more evident in *Super Mario: Odyssey*, where the designers have names and surnames, but it is also the case with matrix multiplication. The advanced methods all depart from the definition, breaking its intended meaning and replacing it with something that has the same behaviour. The geometrical intuition behind the original definition is lost in Strassen's algorithm. Instead, we win new insights: new interpretations stemming from discoveries and inventions alike.

3.3 Why these rules and not others?

The complex web of entangled constraints that defines mathematical practice makes it difficult to understand why mathematics is played according to specific rules and not others. This is precisely the strawman of cultural constructivism. After all, is not everything "made up"?

In its most evident manifestations, rule choice in mathematics appears at the level of accepting or rejecting specific proof methods, such as non-constructive proofs or simply rejecting or embracing specific axioms or assumptions. These are the obvious ones. The more invisible cognitive, social and political constraints also determine the practice and due to their relative invisibility reinforce the idea that the rules for doing mathematics are fixed, immovable and perpetually true.

The comparison with speedrunning can help defeat the strawman of cultural constructivism while supporting the constraint-based account of mathematical practice. Why do runners play according to specific categories? Who moderates, decides and fixes the rules of each category? And, ultimately, why do players engage in these categories at all? Why do runners even practice speedrunning?

The latter question is the easiest to answer: speedrunners run for fun. It might seem naive, but this is the runners' answer. In 2017, David Snyder compiled in a book interviews to ten of the most relevant speedrunners of the time. When asked why they did it, they all replied that it is just fun. Take goatropem, a popular *Halo: Combat Evolved* (Bungie, 2001) runner who says that the game is "really fun, something about the engine feels really good, it's enjoyable to run around, to use weapons, to kill the enemies; it's just fun" (Snyder 2017, 141). For him, speedrunning is "creating new content for this game you already enjoy a lot", "kind of like a DIY DLC[11]" (141).

Surely, sometimes there is money involved, such as with players making a living out of Twitch subscriptions, the platform where they stream their runs. However, as the runners themselves acknowledge, when money is the primary motivation, they are not satisfied with their performance and are more prone to burnout. As *Super Mario 64* runner cheese05 puts it, "you're thinking, what am I speedrunning for right now? For money? Or because I want to speedrun? If you're just doing it for money it's extremely difficult. You always have a bad time. You get angrier more, you have less good runs, it's very stressful." (110). He attributes his love for *Super Mario 64* to its "fast pace and movement"; "it's fun to speedrun, it really is" (104).

At its core, runners find speedrunning fun because they build on an existing video game that is already fun for them. The rules and categories imposed on top are designed to make it even more enjoyable. When digging deeper, it is Hemmingsen's ethos of speedrunning all over again. Players value skill, knowledge and subversion. That is what makes them want to play, and that is what makes them play by the rules they play.

We can use this proposed ethos as a lens to understand why those rules and not others. Take, for example, the case of cheat codes, which are banned in most categories. A cheat code is a specific button input in some specific part of the game that gives the player some advantage. Unlike glitches, the designers intended cheat codes as a secret feature. Hemmingsen presents two examples: the level selection screen in *Sonic: The Hedgehog* (Sonic Team, 1991), allowing the player to skip to the last level, or the special phone numbers in *Grand Theft Auto IV* (Rockstar North, 2008), where dialling them on the in-game phone provides the player with infinite health and ammunition. These are part of the game's code, so executing them does not break the game. However, they are not allowed by the speedrunning community.

Cheat codes attack the very ethos of speedrunning, making the practice uninteresting. There would be no skill needed, no shared knowledge required, and no rebellion happening. No constitutive skill, because choosing a level or having infinite health removes all challenges. No collective knowledge, since cheat codes might be somewhat hidden, but they were introduced by the developers and made public through mainstream media channels, requiring no research, no sense of community, no discovery, and no invention. No subversion, because if the game developers designed it, it was all intended. There is no breaking the rules. It is not fun anymore.

Another example is the glitch hunting community for *The Legend of Zelda: Majora's Mask* (Nintendo, 2000)[12]. For the last 20 years, the community of glitch hunters researched different conjectures to circumvent the game's most significant obstacle: its four compulsory dungeons. In August of 2019, a glitch was discovered that allowed runners to jump directly to the end of the game. It was a debugging screen, left in the Japanese Wii U Virtual Console version. Like cheat codes, it was in the game's code, but the community did not recognise as valid the record-breaking runs performed with that version; it simply was uninteresting. The glitch required no

11. DLC stands for downloadable content published by the developer after the game's launch, to keep users playing the game.

12. The following two paragraphs are based on (Lowest Percent 2021a).

technical skill and removed all challenge from the run. Later that year, a glitch known as Stale Reference Manipulation (SRM) was discovered[13], a bug in how the code handles its memory pointers. SRM allowed players to skip relevant sections of the game, and it required a great deal of mechanical skill to execute. After extensive research and practice, players implemented it in successful record-breaking runs.

In February of 2020, an even bigger exploit was found, known as Arbitrary Code Execution (ACE). This allows the player to execute arbitrary code inside the game, making it relatively easy to skip the four dungeons in the game. Although both the debugging screen and ACE effects are game-breaking, the latter requires skill and knowledge and was unintended. The community accepted SRM and ACE widely, and the new runs were incorporated into the leaderboards.

Setting aside the similarities between mathematical research practices and glitch hunting, it is worth noting that the community's lack of interest in the initial debugging screen skip was due to a lack of fun; a lack of insight into the game's code, a lack of conceptual freedom to tinker with what the game had to offer. Similarly, a mathematician may be disappointed at the perspective of, say, opaque non-constructive proofs. Not due to obscure metaphysical concerns or deep epistemological motivations, but rather because constructive proofs are simply more challenging and fun for them; the alternative removes the challenge, the collective knowledge, and the subversion.

Ultimately, these similarities carry over to the domain of tools used and allowed by the community. Speedrunning revolves around video games, which require some form of electronic hardware to be executed. Hence, the runner's first choice comes in the form of what hardware combination to choose: what controller, what console, what screen... Since different consoles have different computational processing power, a game in the original version will likely run slower than in a newer system. The community often forces players to run in the original system, and they do not allow any hardware modifications. Sometimes there are further restrictions imposed, regarding whether the version of the game is American (NTSC) or European (PAL), which can affect the time of the run to the order of seconds[14]. The desire behind this is again to preserve skill: all runners should compete under the same conditions, with an aim to runs —like proofs— begin replicable.

There is also an issue of surveyability, tightly related to the role of computer-assisted runs. Speedrunners often use emulators of the games to pause and move frame by frame. In this way, glitch hunters can inspect the game's behaviour to every possible input, seeing how the memory's state changes and the game reacts. These are called Tool-Assisted Speedruns (TAS) and can be used to craft near-optimal routes that are impossible to execute for a human player. TAS are not a legitimate form of competition and are used for research purposes only.

There is debate in the community as to the role TAS should play, though arguably it is less heated than the one surrounding proof-assistants in mathematics. The issue at core is that in both cases, the proofs being produced (whether it is a route's viability or a theorem that is being

13. Technically, examples of SRM had been observed as early as 2009, but no coherent explanation was available.

14. This is due to the original difference between refresh rates in American and European televisions, the latter refreshing at 50 Hz and the American at 60 Hz. Accordingly, the European versions of the games move slower since each frame stays longer on the screen, giving the player more time to react. There exist conversion factors that allow the community to standardise running times between such different versions of a game.

proven) are no longer surveyable by humans. The video footage of a TAS can be heavily altered since it moves so fast that detecting cuts requires close inspection. Similarly, long computer-verified proofs are mostly unreadable to the human mathematician, and confidence in the result is put instead on the programmers of the system at hand.

There is, however, a difference in how the speedrunning community and the mathematical community approach the debate around computer tools. This is because speedrunners accept its research's empirical nature, based on trial and error against the code. In this way, a TAS is just a microscope that inspects the game's behaviour one frame at a time. For mathematicians, however, formal proofs are supposed to be the proper object of interest, claiming their informal natural language proofs to be just convincing arguments for the existence of formal counterparts.

However, as pointed out by Lakatos, mathematics is quasi-empirical rather than Euclidean (Lakatos 1998). Proof-assistants are thus a distorted model of mathematical practice (De Millo, Lipton, and Perlis 1979), in the same way that a TAS is not an accurate depiction of the possibilities of a human player. Unlike mathematics, speedrunners accept TAS as a tool to understand the game better, not as the ultimate goal —since it is simply impossible to achieve and, more importantly, not that fun.

Hence, when the speedrunning communities choose their rules and the role they give to auxiliary tools, decisions are informed by fun. Players want the runs to be interesting, insightful and challenging. Whatever tools contribute to interesting skill-based challenges, illuminating research findings, and subversive play will all be welcome. Nobody thinks optimal speedruns exist in some Platonic world of ideas, waiting to be executed in a tool-assisted run. Speedrunning is quasi-empirical in the Lakatosian sense and lies in the realm of the analytic a posteriori, the same place that Wagner gives to mathematics. The code is there, but you must execute it. The game must be played.

4 Conclusion

This paper has compared mathematical practice to speedrunning in video games, showing how similar they are both at a general level and at the level of specific episodes of the history of mathematics and speedrunning. The goal has been to support a constraint-based account of mathematical practice, such as the one posited by Wagner. The claim is that mathematics is an activity humans carry out constrained by simultaneous and changing social and cognitive constraints. In this way, a set of human-made rules delimits a space of possibilities where mathematical discovery occurs.

The analogy with video game speedrunning shows that there is nothing intrinsically otherworldly about the rules used by mathematicians, despite their incredible success at describing nature or organising society. Video games are also a system of hardcoded rules made up by human agents that nevertheless contain secrets, glitches, and exploits to be found by the speedrunning community. The analogy showcases that, in fact, it is the rules added on top of the mechanics that make for an exciting practice.

Due to space constraints, two interesting issues have been left out of this work. Firstly, a close-up study of how glitch hunting and mathematical research resemble each other in methodology, language, and their quasi-empirical character. Secondly, the proper mathematical work on speedrunning, such as (Bosboom et al. 2015; Lafond 2018; Coulombe and Lynch 2020; Frei, Rossmanith, and Wehner 2020). Most of that research refers to the computational complexity of finding optimal runs, the parameterised complexity of tool-assisted speedruns or the computability of deciding the existence of a winning strategy in a suitable generalisation of popular video games. Whether these results have any meaningful impact on glitch hunting is unclear, but they pose the question of whether speedrunning is just mathematics in disguise. This would make for an interesting case study, applying ideas of dismotivation, formalisation and arbitration devised by Wagner to explain what is considered mathematics proper.

Finally, it is worth noting that much of the theoretical framework used for this research comes from game studies, showing that broadening the scope of the literature beyond historical and philosophical texts on mathematics can help the philosophy of mathematical practice. Comparison with other practices can be both helpful and enlightening.

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